

# G-FANO THREEFOLDS, I

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ABSTRACT. We classify Fano threefolds with only terminal singularities whose canonical class is Cartier and divisible by 2, and satisfying an additional assumption that the  $G$ -invariant part of the Weil divisor class group is of rank 1 with respect to an action of some group  $G$ . In particular, we find a lot of examples of Fano 3-folds with “many” symmetries.

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## 1. INTRODUCTION.

**1.1.** Let  $Y$  be an algebraic variety  $X$  over a field  $\mathbb{k}$  and let  $G$  be a group. Following works of Yu. I. Manin [Man67] we say that  $X$  is a  $G$ -variety if the group  $G$  acts on  $\bar{X} := X \otimes_{\mathbb{k}} \bar{\mathbb{k}}$ , where  $\bar{\mathbb{k}}$  is the algebraic closure of  $\mathbb{k}$ . Moreover, we assume that  $X$ ,  $G$  and  $\mathbb{k}$  satisfy one of the following two conditions.

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(a) *Algebraic case.*  $G$  is the Galois group of  $\bar{\mathbb{k}}$  over  $\mathbb{k}$  acting on  $\bar{X}$  through the second factor. The action of  $G$  on  $X$  is trivial.

(b) *Geometric case.* The field  $\mathbb{k}$  is algebraically closed,  $G$  is a finite group, and the action of  $G$  on  $X$  is given by a homomorphism  $G \rightarrow \text{Aut}_{\mathbb{k}}(X)$ .

A  $G$ -morphism (resp. rational  $G$ -map) of  $G$ -varieties is a  $\mathbb{k}$ -morphism (resp.  $\mathbb{k}$ -rational map) commuting with the action of  $G$  in the geometric case. A projective  $G$ -morphism  $f : X \rightarrow Z$  of normal  $G$ -varieties is called  *$G$ -Mori fiber space* if  $X$  has at worst terminal  $G\mathbb{Q}$ -factorial singularities (see 2.2),  $f_*\mathcal{O}_X = \mathcal{O}_Z$ , the relative invariant Picard group  $\text{Pic}(X/Z)^G$  is of rank 1, and the anticanonical divisor  $-K_X$  is  $f$ -ample. In the particular case where  $\dim Z = 0$ ,  $X$  is called a  *$G\mathbb{Q}$ -Fano variety*. If furthermore the canonical divisor is Cartier, then we say that  $X$  is  *$G$ -Fano variety*.

Throughout this paper we assume that the ground field has characteristic 0. The following is an easy consequence of the Minimal Model Program [Mor88, 0.3.14] (cf. [Pro09, 4.2]).

**1.2. Proposition.** *Let  $V$  be a  $G$ -variety of dimension  $\leq 3$ . The following are equivalent:*

- (i)  $\kappa(V) = -\infty$ ,
- (ii)  $V$  is geometrically uniruled,
- (iii)  $V$  is  $G$ -birationally isomorphic to a variety  $X$  having a structure of  $G$ -Mori fiber space.

Birational classification of  $G$ -surfaces is developed very well [Man67], [Isk80b]. In this and subsequent papers we consider  $G$ -Mori fiber spaces  $X \rightarrow Z$ , where  $\dim X = 3$  and  $Z$  is a point, i.e. the case of  $G\mathbb{Q}$ -Fano threefolds.

**1.3.** Let  $X$  be a  $G$ -Fano threefold. It is well-known that  $\text{Pic}(X)$  is a finitely generated torsion free abelian group (see, e.g. [IP99, Prop. 2.1.2]). Consider the following composed object:

$$V(X) = (\text{Cl}(X), \text{Pic}(X), K_X, (\ , \ , \ )),$$

where  $\text{Pic}(X)$  is regarded as a sublattice of  $\text{Cl}(X)$ ,  $K_X \in \text{Pic}(X)$  is the canonical class of  $X$ , and  $(\ , \ , \ )$  is the intersection form  $\text{Pic}(X) \times \text{Pic}(X) \times \text{Cl}(X) \rightarrow \mathbb{Z}$ . Since the singularities of  $X$  are isolated cDV [Rei87],  $\text{Pic}(X)$  is a primitive sublattice in  $\text{Cl}(X)$ , i.e. the quotient  $\text{Cl}(X)/\text{Pic}(X)$  is torsion free [Kaw88, 5.1]. Moreover, since  $\rho(X)^G = 1$ , we have

$$(1.3.1) \quad \text{Cl}(X)^G \text{ is a subgroup of rank 1 containing } K_X.$$

**1.4.** In this paper we give a classification of one class of Gorenstein  $G$ -Fano threefolds. More precisely, we consider Fano threefolds such

that  $-K_X = 2S$  for some ample Cartier divisor  $S$ . Then  $X$  is called a *del Pezzo threefold* (see 3.1). Smooth del Pezzo threefolds were classified by Iskovskikh [Isk80a], see also [Fuj84], [IP99]. Singular ones were discussed from different points of view in many works [Fuj86], [Shi89], [Fuj90], [CJR08], [JP08]. We are interested basically in group actions on terminal del Pezzo threefolds  $X$  and the structure of the lattice  $\text{Cl}(X)$ .

**1.5.** Let  $S$  be a smooth del Pezzo surface of degree  $d := K_S^2$ . Then we have  $\text{Pic}(S) = \mathbb{Z}^{10-d}$ . Define

$$\Delta := \{\alpha \in \text{Pic}(S) \mid \alpha^2 = -2, \quad \alpha \cdot K_S = 0\}.$$

Then  $\Delta$  is a root system in  $(K_S)^\perp \otimes \mathbb{R}$ . Depending on  $d$ ,  $\Delta$  is of the following type ([Man86]):

| d        | 1     | 2     | 3     | 4     | 5     | 6                | 7   | 8'    | 8'' |
|----------|-------|-------|-------|-------|-------|------------------|-----|-------|-----|
| $\Delta$ | $E_8$ | $E_7$ | $E_6$ | $D_5$ | $A_4$ | $A_1 \times A_2$ | $-$ | $A_1$ | $-$ |

where  $8'$  (resp.  $8''$ ) corresponds to  $\mathbb{P}^1 \times \mathbb{P}^1$  (resp. Hirzebruch surface  $\mathbb{F}_1$ ).

**1.6.** Now let  $X$  be a del Pezzo threefold. Let  $S \in |-\frac{1}{2}K_X|$  be a smooth member [Shi89] and let  $\iota : S \hookrightarrow X$  be the natural embedding. Then  $S$  is a del Pezzo surface of degree  $d = -\frac{1}{8}K_X^3$ . It is easy to show that the restriction map  $\iota^* : \text{Cl}(X) \rightarrow \text{Pic}(S)$  is injective and its cokernel is torsion free (see 3.9.3). Define the following subsets in  $\Delta \subset \text{Pic}(S)$ :

$$\Delta' := \{\alpha \in \text{Pic}(S) \mid \alpha^2 = -2, \quad \alpha \cdot K_S = \alpha \cdot \iota^* \text{Cl}(X) = 0\},$$

$$\Delta'' := \{\alpha \in \iota^* \text{Cl}(X) \mid \alpha^2 = -2, \quad \alpha \cdot K_S = 0\}.$$

In other words,

$$\Delta' = \Delta \cap (\iota^* \text{Cl}(X))^\perp, \quad \Delta'' = \Delta \cap \iota^* \text{Cl}(X).$$

If  $\Delta'$  (resp.  $\Delta''$ ) is non-empty, then it is a root subsystem in  $\Delta$ . Assume that  $X$  is a  $G$ -variety. Then the group  $G$  naturally acts on  $\iota^* \text{Cl}(X)$  and  $\Delta''$  preserving the class of  $K_S$  and the intersection form.

Our classification of  $G$ -del Pezzo threefolds is by types of root systems  $\Delta'$  and  $\Delta''$ .

**1.7. Theorem.** *Let  $X$  be a  $G$ -del Pezzo threefold and let  $d(X) := -\frac{1}{8}K_X^3$ . There are the following possibilities:*

|            | r | $X$      | $\bar{X}$      | $Z$                | $\Delta'$ | $\Delta''$       | p   | s                   |
|------------|---|----------|----------------|--------------------|-----------|------------------|-----|---------------------|
| $d(X) = 1$ |   |          |                |                    |           |                  |     |                     |
| $1^\circ$  | 1 | $V_1$    | —              | pt                 | $E_8$     | —                | 0   | $21 - h$            |
| $2^\circ$  | 2 | (5.2.6)  | —              | $\mathbb{P}^1$     | $D_7$     | —                | 0   | $22 - h$            |
| $3^\circ$  | 2 | (5.2.1)  | —              | $\mathbb{P}^2$     | $A_7$     | —                | 0   | 22                  |
| $4^\circ$  | 2 | (5.2.12) | $V_2$          | pt                 | $E_7$     | $A_1$            | 2   | $22 - h, h \leq 10$ |
| $5^\circ$  | 3 |          | (5.2.7)        | $\mathbb{P}^1$     | $D_6$     | $2A_1$           | 4   | $23 - h$            |
| $6^\circ$  | 3 |          | (5.2.2)        | $\mathbb{P}^2$     | $A_6$     | $A_1$            | 2   | 23                  |
| $7^\circ$  | 3 |          | $V_3$          | pt                 | $E_6$     | $A_2$            | 6   | $23 - h$            |
| $8^\circ$  | 4 |          | (5.2.3)        | $\mathbb{P}^2$     | $A_5$     | $A_1 \times A_2$ | 8   | 24                  |
| $9^\circ$  | 4 |          | $V_4$          | pt                 | $D_5$     | $A_3$            | 12  | $24 - h, h \leq 2$  |
| $10^\circ$ | 5 |          | (5.2.8)        | $\mathbb{P}^1$     | $D_4$     | $D_4$            | 24  | $25 - h$            |
| $11^\circ$ | 5 |          | $V_5$          | pt                 | $A_4$     | $A_4$            | 20  | 25                  |
| $12^\circ$ | 6 |          | (5.2.5)        | $\mathbb{P}^2$     | $A_3$     | $D_5$            | 40  | 26                  |
| $13^\circ$ | 7 | 8.1      | $V_6$          | $\mathbb{P}^2$     | $A_2$     | $E_6$            | 72  | 27                  |
| $14^\circ$ | 8 | 7.8      | $\mathbb{P}^3$ | pt                 | $A_1$     | $E_7$            | 126 | 28                  |
| $d(X) = 2$ |   |          |                |                    |           |                  |     |                     |
| $15^\circ$ | 1 | $V_2$    | —              | pt                 | $E_7$     | —                | 0   | $10 - h$            |
| $16^\circ$ | 2 | (5.2.7)  | —              | $\mathbb{P}^1$     | $D_6$     | $A_1$            | 0   | $11 - h$            |
| $17^\circ$ | 2 | (5.2.2)  | —              | $\mathbb{P}^2$     | $A_6$     | —                | 0   | 11                  |
| $18^\circ$ | 2 | (5.2.13) | $V_3$          | pt                 | $E_6$     | —                | 2   | $11 - h, h \leq 5$  |
| $19^\circ$ | 3 | 4.2.1    | —              | $(\mathbb{P}^1)^2$ | $A_5$     | $A_2$            | 0   | 12                  |
| $20^\circ$ | 3 |          | (5.2.3)        | $\mathbb{P}^2$     | $A_5$     | $A_1$            | 2   | 12                  |
| $21^\circ$ | 3 |          | $V_4$          | pt                 | $D_5$     | $A_1$            | 4   | $12 - h, h \leq 2$  |
| $22^\circ$ | 4 |          | (5.2.8)        | $\mathbb{P}^1$     | $D_4$     | $3A_1$           | 8   | $13 - h$            |
| $23^\circ$ | 4 |          | $V_5$          | pt                 | $A_4$     | $A_2$            | 6   | 13                  |
| $24^\circ$ | 5 |          | (5.2.5)        | $\mathbb{P}^2$     | $A_3$     | $A_1 \times A_3$ | 12  | 14                  |
| $25^\circ$ | 6 | 8.1      | $V_6$          | $\mathbb{P}^2$     | $A_2$     | $D_5$            | 20  | 15                  |
| $26^\circ$ | 7 | 7.7      | $\mathbb{P}^3$ | pt                 | $A_1$     | $D_6$            | 32  | 16                  |
| $d(X) = 3$ |   |          |                |                    |           |                  |     |                     |
| $27^\circ$ | 1 | $V_3$    | —              | pt                 | $E_6$     | —                | 0   | $\leq 5$            |
| $28^\circ$ | 2 | (5.2.3)  | —              | $\mathbb{P}^2$     | $A_5$     | $A_1$            | 0   | 6                   |
| $29^\circ$ | 3 | 9.4      | (5.2.8)        | $\mathbb{P}^1$     | $D_4$     | —                | 3   | $6 \leq s \leq 7$   |
| $30^\circ$ | 5 | 8.1      | $V_6$          | $\mathbb{P}^2$     | $A_2$     | $2A_2$           | 9   | 9                   |
| $31^\circ$ | 6 | 7.6      | $\mathbb{P}^3$ | pt                 | $A_1$     | $A_5$            | 15  | 10                  |

|            | r | $X$                | $\bar{X}$      | $Z$                | $\Delta'$ | $\Delta''$       | p | s                 |
|------------|---|--------------------|----------------|--------------------|-----------|------------------|---|-------------------|
| $d(X) = 4$ |   |                    |                |                    |           |                  |   |                   |
| $32^\circ$ | 1 | $V_4$              | —              | pt                 | $D_5$     | —                | 0 | $\leq 2$          |
| $33^\circ$ | 2 | (5.2.8)            | —              | $\mathbb{P}^1$     | $D_4$     | —                | 0 | $1 \leq s \leq 3$ |
| $34^\circ$ | 3 | 4.2.2              | —              | $(\mathbb{P}^1)^2$ | $A_3$     | $2A_1$           | 0 | 4                 |
| $35^\circ$ | 4 | 8.1                | $V_6$          | $\mathbb{P}^2$     | $A_2$     | $2A_1$           | 4 | 5                 |
| $36^\circ$ | 5 | 7.5                | $\mathbb{P}^3$ | pt                 | $A_1$     | $A_1 \times A_3$ | 8 | 6                 |
| $d(X) = 5$ |   |                    |                |                    |           |                  |   |                   |
| $37^\circ$ | 1 | $V_5$              | —              | pt                 | $A_4$     | —                | 0 | 0                 |
| $d(X) = 6$ |   |                    |                |                    |           |                  |   |                   |
| $38^\circ$ | 2 | $V_6$              | —              | $\mathbb{P}^2$     | $A_2$     | $A_1$            | 0 | 0                 |
| $39^\circ$ | 3 | $(\mathbb{P}^1)^3$ | —              | pt                 | $A_1$     | $A_2$            | 0 | 0                 |
| $d(X) = 8$ |   |                    |                |                    |           |                  |   |                   |
| $40^\circ$ | 1 | $\mathbb{P}^3$     | —              | pt                 | —         | —                | 0 | 0                 |

Here  $\bar{X}/Z$  is a primitive birational model of  $X$  (see Theorem 3.9) and  $h := h^{1,2}(\hat{X})$ , where  $\hat{X}$  is the standard resolution of  $X$ . For compactness, we denote  $(\mathbb{P}^1)^k := \underbrace{\mathbb{P}^1 \times \cdots \times \mathbb{P}^1}_k$ . For other notation we refer to 2.1.

**1.8. Remark.** For  $d(X) \leq 2$  any del Pezzo threefold automatically has  $G$ -structure (see Remark 3.4.1). So, in this case,  $1^\circ - 26^\circ$  is a complete list of del Pezzo threefolds with  $d(X) \leq 2$ .

**1.9. Remark.** Singular three-dimensional cubics (without group action) whose singularities are only *nodes* and their small resolutions were classified in [FW]. There is the following correspondence between our list and the classification in [FW]:  $31^\circ \longleftrightarrow \text{J15}$ ,  $30^\circ \longleftrightarrow \text{J14}$ ,  $29^\circ \longleftrightarrow \text{J11}$ ,  $28^\circ \longleftrightarrow \text{J9}$ ,  $27^\circ \longleftrightarrow \text{J1-J5}$ .

We hope that our result can be useful for applications to the classification of finite subgroups of the Cremona group  $\text{Cr}_3(\mathbb{k})$  [Pro09], [Pro10], and also the birational classification of rational algebraic threefolds over non-closed fields (cf. [Man67]).

The paper is organized as follows. In Sections 2 and 3 we collect some known results. In Sections 4 and 5 we classify primitive del Pezzo threefolds with  $\text{rk Cl}(X) = 3$  and 2, respectively. The results of §5 were known earlier [JP08]. We give a short proof for the convenience of the reader. Section 6 describes root systems  $\Delta'$  on del Pezzo threefolds. In Sections 7 and 8 we classify del Pezzo threefolds with  $\text{rk Cl}(X) \geq 8 - d$ , where  $d$  is the (half-canonical) degree of  $X$ . Section 9 is devoted to the proof of Theorem 1.7.

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## 2. PRELIMINARIES.

**2.1. Notation.** We work over an algebraically closed field of characteristic 0. Throughout this paper  $X$  denotes a del Pezzo threefold with at worst terminal Gorenstein singularities. Thus we can write  $-K_X = 2S$ , where  $S = S_X$  is a ample Cartier divisor of  $S$  defined up to linear equivalence. Everywhere below we use the following notation:

$\rho = \rho(X) := \text{rk Pic}(X)$ ;  
 $\text{Cl}(X)$  is the Weil divisor class group;  
 $r = r(X) := \text{rk Cl}(X)$ ;  
 $d = d(X) := S^3 = -K_X^3/8$ , the degree of  $X$ ;  
 $p = p(X)$  is the number of planes on  $X$ ;  
 $s = s(X)$  is the number of singular points of  $X$  under an additional assumption that  $X$  has at worst nodes;  
 $V_6 \subset \mathbb{P}^7$  is a *smooth* del Pezzo threefold with  $d(X) = 6$  and  $\rho = 2$ , see Theorem 3.5;  
 $V_5 \subset \mathbb{P}^6$  is a *smooth* del Pezzo threefold of degree 5 (see [IP99]);  
 $V_d$ , for  $d = 1, \dots, 4$ , is a del Pezzo threefold of degree  $d$  with terminal *factorial* singularities (see Theorem 3.4).

**2.2. Terminal singularities (see [Rei87]).** Let  $(X, P)$  be a germ of a three-dimensional terminal singularity. Then  $(X, P)$  is isolated, i.e.,  $\text{Sing}(X) = \{P\}$ . The *index* of  $(X, P)$  is the minimal positive integer  $r$  such that  $rK_X$  is Cartier. If  $r = 1$ , then  $(X, P)$  is Gorenstein. In this case  $(X, P)$  is analytically isomorphic to a hypersurface singularity of multiplicity 2.

Let  $X$  be a threefold with Gorenstein terminal singularities. Then any Weil  $\mathbb{Q}$ -Cartier divisor is Cartier (see e.g. [Kaw88, Lemma 5.1]). Equivalently,  $\text{Pic}(X)$  is a primitive sublattice in  $\text{Cl}(X)$ .

Let  $X$  be a  $G$ -variety. We say that  $X$  has only  $G\mathbb{Q}$ -factorial singularities if any  $G$ -invariant Weil divisor is  $\mathbb{Q}$ -Cartier.

**2.2.1. Theorem-Definition ([Kaw88, Corollary 4.5]).** *Let  $X$  be a threefold with terminal singularities. Then there exists a projective birational morphism  $\xi : \hat{X} \rightarrow X$  such that*

- (i)  $\hat{X}$  is normal and has only terminal  $\mathbb{Q}$ -factorial singularities;

- (ii)  $\xi$  is a crepant morphism, that is,  $K_{\hat{X}} = \xi^* K_X$ ;
- (iii)  $\xi$  is small, that is, its exceptional locus does not contain any divisors.

Such a morphism is called  $\mathbb{Q}$ -factorialization of  $X$ . Any two  $\mathbb{Q}$ -factorializations of  $X$  are connected by a sequence of flops.

**2.2.2. Theorem** [Cut88]. *Let  $X$  be a rationally connected threefold with terminal factorial singularities. Assume that  $-K_X = 2S$  for some divisor  $S$  and  $\rho(X) > 1$ . Let  $f : X \rightarrow Z$  be an extremal  $K_X$ -negative contraction. Then one of the following holds:*

- (i)  $Z \simeq \mathbb{P}^1$  and  $f$  is a quadric bundle, i.e. there is an embedding  $X \hookrightarrow \mathbb{P}(\mathcal{E})$ , where  $\mathcal{E}$  is a rank 4 vector bundle on  $Z$ , so that each fiber of  $f$  is a quadric in the fiber of  $\mathbb{P}(\mathcal{E})/Z$ ;
- (ii)  $X$  is smooth,  $Z$  is a smooth rational surface, and  $X = \mathbb{P}(\mathcal{E})$ , where  $\mathcal{E}$  is a rank 2 vector bundle on  $Z$ ;
- (iii)  $Z$  is a threefold with terminal factorial singularities and  $f$  is blowup of a smooth point on  $Z$ .

### 3. GENERALITIES ON DEL PEZZO THREEFOLDS

**3.1. Definition.** Let  $X$  be a projective variety  $X$  with at worst terminal Gorenstein singularities.\* We say that  $X$  is a *del Pezzo threefold* (resp. *weak del Pezzo threefold*) if its anti-canonical class  $-K_X$  is divisible by 2 and is ample (resp. nef and big).

Note that if  $X$  is a Fano threefold with at worst terminal Gorenstein singularities such that  $-K_X$  is divisible by some positive integer  $q$ , then  $q \leq 4$ . Moreover,  $q = 4$  iff  $X \simeq \mathbb{P}^3$  and  $q = 3$  iff  $X$  is a quadric in  $\mathbb{P}^4$  (see e.g. [IP99, Th. 3.1.14]). Thus, for a del Pezzo threefold  $X$  such that  $X \not\simeq \mathbb{P}^3$ , the divisor  $-\frac{1}{2}K_X$  is a primitive element of the lattice  $\text{Cl}(X)$ .

**3.2. Theorem** ([Fuj86]). *Let  $X$  be a del Pezzo threefold. Then  $d(X) \leq 8$ . Moreover, if  $d(X) = 8$ , then  $X \simeq \mathbb{P}^3$ . If  $d(X) = 7$ , then  $X \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(1))$ . If  $d(X) = 6$ , then  $\rho(X) = 2$  or  $3$ . If  $d(X) \leq 5$ , then  $\rho(X) = 1$ .*

**3.3. Lemma.** *Let  $X$  be a del Pezzo threefold. If  $X$  is factorial and singular, then  $\rho(X) = 1$ .*

*Proof.* Assume that  $\rho(X) > 1$ . Let  $f_i : X \rightarrow Z$  be all extremal contractions. By Theorem 2.2.2, since  $X$  is singular,  $\dim Z_i \neq 2$  and each  $f_i$  has a two-dimensional fiber  $F_i$ . Then, for  $i \neq j$  fibers  $F_i$  and  $F_j$  do not meet each other. Therefore, all the contractions  $f_i$  are birational and

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\*In papers [Fuj90], [CJR08] authors considered del Pezzo varieties whose singularities are more general than terminal.

$F_i$  are exceptional divisors. But this is impossible (see e.g. [IP99, Proof of Th. 7.1.6]).  $\square$

**3.4. Theorem** ([Isk80a], [Shi89], [Fuj84], [Fuj86], [Fuj90]). *Let  $X$  be a del Pezzo threefold and let  $S = -\frac{1}{2}K_X$ .*

- (i)  $\dim |S| = d(X) + 1$ .
- (ii) *The linear system  $|S|$  is base point free (resp. very ample) for  $d(X) \geq 2$  (resp.  $d(X) \geq 3$ ). If  $d(X) \geq 4$ , then the image of  $X_{d(X)} \subset \mathbb{P}^{d(X)+1}$  of  $X$  under the embedding given by  $|S|$  is an intersection of quadrics.*
- (iii) *If  $d(X) = 1$ , then the linear system  $|S|$  has a unique base point which is a smooth point of  $X$ . In this case  $|S|$  defines a rational map  $X \dashrightarrow \mathbb{P}^2$  whose general fiber is an elliptic curve. The variety  $X$  is isomorphic to a hypersurface of degree 6 in the weighted projective space  $\mathbb{P}(1^3, 2, 3)$ .*
- (iv) *If  $d(X) = 2$ , then  $|S|$  defines a double cover  $X \rightarrow \mathbb{P}^3$  whose branch locus  $B \subset \mathbb{P}^3$  is a surface of degree 4 with at worst isolated singularities. The variety  $X$  is isomorphic to a hypersurface of degree 4 in  $\mathbb{P}(1^4, 2)$ .*
- (v) *If  $d(X) = 3$ , then  $X$  is isomorphic to a cubic in  $\mathbb{P}^4$ .*
- (vi) *If  $d(X) = 4$ , then  $X$  is isomorphic to a complete intersection of two quadrics in  $\mathbb{P}^5$ .*

The del Pezzo threefolds with  $d(X) = 1$  and  $d(X) = 2$  have their names: *double Veronese cone* and *quartic double solid*, respectively.

**3.4.1. Remark.** Let  $X$  be a del Pezzo threefold of degree 1 (resp. 2). Then there is a finite of degree 2 morphism  $\varphi : X \rightarrow \mathbb{P}(1^3, 2)$  (resp.  $\varphi : X \rightarrow \mathbb{P}^3$ ). The corresponding natural Galois involution  $X \rightarrow X$  is called *Bertini* (resp. *Geiser*) involution. Therefore, any del Pezzo threefold  $X$  with  $d(X) \leq 2$  is a  $G$ -del Pezzo (in the geometric sense).

**3.5. Theorem** ([Fuj84], [IP99]). *Let  $X$  be a smooth del Pezzo threefold with  $d(X) = 6$ . Then either  $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  or  $X \simeq V_6 \subset \mathbb{P}^7$ , where  $V_6$  is unique up to isomorphism and is isomorphic to a divisor of bidegree  $(1, 1)$  in  $\mathbb{P}^2 \times \mathbb{P}^2$ .*

**3.6. Proposition** (see [IP99]). *Let  $X$  be a smooth del Pezzo threefold. Then the Hodge number  $h^{1,2}(X)$  is given by the following table:*

|              |    |    |   |   |   |   |   |   |
|--------------|----|----|---|---|---|---|---|---|
| $d(X)$       | 1  | 2  | 3 | 4 | 5 | 6 | 7 | 8 |
| $h^{1,2}(X)$ | 21 | 10 | 5 | 2 | 0 | 0 | 0 | 0 |



**3.7. Definition.** Let  $X$  be a weak del Pezzo threefold and let  $S = -\frac{1}{2}K_X$ . An irreducible surface  $\Pi \subset X$  is called a *plane* if  $S^2 \cdot \Pi = 1$  and, in case  $d(X) = 1$ , the base point of  $|S|$  does not lie on  $\Pi$ .

**3.7.1. Lemma.** *Let  $X$  be a del Pezzo threefold. If  $\Pi \subset X$  is a plane, then  $\Pi \simeq \mathbb{P}^2$  and  $\mathcal{O}_\Pi(S) = \mathcal{O}_{\mathbb{P}^2}(1)$ .*

*Proof.* The statement is obvious if  $d(X) \geq 3$  because the divisor  $S$  is very ample in this case. If  $d(X) = 2$ , then  $|S|$  defines a double cover  $\varphi : X \rightarrow \mathbb{P}^3$  so that  $\varphi(\Pi)$  is a projective plane on  $\mathbb{P}^3$ . Thus  $\varphi|_\Pi : \Pi \rightarrow \varphi(\Pi) \simeq \mathbb{P}^2$  is a finite birational morphism, so it is an isomorphism. Finally if  $d(X) = 1$ , then  $|S|$  defines a rational map  $\varphi : X \dashrightarrow \mathbb{P}^2$  so that its restriction to  $\Pi$  is a morphism which must be finite and birational. As above we get  $\Pi \simeq \mathbb{P}^2$ .  $\square$

**3.7.2. Lemma.** *If  $\Pi \subset X$  is a plane, then there is a  $\mathbb{Q}$ -factorialization  $\xi : \hat{X} \rightarrow X$  such that for the proper transform  $\hat{\Pi}$  of  $\Pi$  we have  $\hat{\Pi} \simeq \mathbb{P}^2$  and  $\mathcal{O}_{\hat{\Pi}}(\hat{\Pi}) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$ . Therefore,  $\hat{\Pi}$  is contractible, i.e. there is a birational contraction  $\hat{X} \rightarrow X'$  of  $\hat{\Pi}$  to a smooth point. Conversely, if  $\xi : \hat{X} \rightarrow X$  is a  $\mathbb{Q}$ -factorialization and  $\hat{\Pi} \subset \hat{X}$  is an irreducible surface such that  $\hat{\Pi} \simeq \mathbb{P}^2$  and  $\mathcal{O}_{\hat{\Pi}}(\hat{\Pi}) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$ , then  $f(\hat{\Pi})$  is a plane on  $X$ .*

*Proof.* Let  $\Pi \subset X$  be a plane. Take a  $\mathbb{Q}$ -factorialization  $\xi : \hat{X} \rightarrow X$  so that  $\hat{\Pi}$  is  $f$ -nef. One can do it by performing flops over  $X$ . Assume that  $\hat{\Pi}$  is nef. Then by the base point free theorem the linear system  $|n\hat{\Pi}|$  is base point free for  $n \gg 0$ . Hence  $|n\Pi|$  has no fixed components. Since  $X$  has at worst isolated singularities, by adjunction we have

$$K_\Pi = (-2S + \Pi)|_\Pi \geq -2S|_\Pi,$$

a contradiction.

Thus  $\hat{\Pi}$  is not nef. Then there is a  $K_{\hat{X}}$ -negative extremal ray  $R$  such that  $\hat{\Pi} \cdot R < 0$ . Since  $K_{\hat{X}}$  is divisible by 2, from the classification of extremal rays (Theorem 2.2.2) we see that  $\hat{\Pi}$  is contractible to a smooth point,  $\hat{\Pi} \simeq \mathbb{P}^2$  and  $\mathcal{O}_{\hat{\Pi}}(\hat{\Pi}) \simeq \mathcal{O}_{\mathbb{P}^2}(-1)$ . The converse statement is obvious.  $\square$

**3.7.3. Lemma.** *Let  $X$  be a del Pezzo threefold and let  $S \in |-\frac{1}{2}K_X|$  be a smooth member. Let  $l \in \text{Pic}(S)$  be an element such that  $l^2 = l \cdot K_S = -1$  (the class of a line  $L \subset S$ ). Assume that  $l \in \iota^* \text{Cl}(X)$ , where  $\iota : S \hookrightarrow X$  is the embedding. Then there exists a unique plane  $\Pi \subset X$  such that  $\iota^*\Pi = l$  (i.e.,  $\Pi \cap S = L$ ).*

*Proof.* Denote by  $\Pi$  any divisor whose class coincides with  $\iota^*l$ . Let  $\xi : \hat{X} \rightarrow X$  be a  $\mathbb{Q}$ -factorialization as in Lemma 3.7.2, let  $\hat{S} := \xi^{-1}(S)$ , and let  $\hat{\Pi}$  be the proper transform of  $\Pi$ . By Shokurov's adjunction

theorem the pair  $(\hat{X}, \hat{\Pi})$  is purely log terminal (PLT). Hence, by the Kawamata-Viehweg vanishing [Fuk97, Prop. 1]

$$H^1(\hat{X}, \mathcal{O}_{\hat{X}}(\hat{\Pi} - \hat{S})) = H^1(\hat{X}, \mathcal{O}_{\hat{X}}(\hat{S} + K_{\hat{X}} + \hat{\Pi})) = 0.$$

Then one can see from the exact sequence

$$0 \longrightarrow \mathcal{O}_{\hat{X}}(\hat{\Pi} - \hat{S}) \longrightarrow \mathcal{O}_{\hat{X}}(\hat{\Pi}) \longrightarrow \mathcal{O}_{\hat{S}}(\iota^* l) \longrightarrow 0$$

that  $H^0(\hat{X}, \mathcal{O}_{\hat{X}}(\hat{\Pi})) \neq 0$ , so we may assume that both  $\hat{\Pi}$  and  $\Pi$  are effective. Since  $S^2 \cdot \Pi = 1$ ,  $\Pi$  is a plane. Finally, if there is another plane  $\Pi'$  such that  $\iota^* \Pi' = l$ , then  $\Pi \sim \Pi'$  and  $\mathcal{O}_{\hat{\Pi}}(\hat{\Pi}) = \mathcal{O}_{\hat{\Pi}}(\hat{\Pi}')$  is positive, a contradiction.  $\square$

**3.7.4. Definition.** We say that a del Pezzo threefold  $X$  is *imprimitive* if it contains at least one plane. Otherwise we say that  $X$  is *primitive*.

The following two theorems are easy consequences of [CJR08, Prop. 2.8].

**3.8. Theorem.** *Let  $X$  be a primitive weak del Pezzo threefold with at worst terminal Gorenstein singularities. Let  $\xi : \hat{X} \rightarrow X$  be a  $\mathbb{Q}$ -factorialization. Then there exists a  $K_{\hat{X}}$ -negative Mori contraction  $f : \hat{X} \rightarrow Z$  such that one of the following holds:*

- (i)  $Z$  is a point,  $\rho(\hat{X}) = 1$ ,  $X$  is factorial, and  $\xi$  is an isomorphism;
- (ii)  $Z \simeq \mathbb{P}^2$ ,  $\rho(\hat{X}) = 2$ , and  $f$  is a  $\mathbb{P}^1$ -bundle, i.e.  $\hat{X}$  is smooth and  $\hat{X} = \mathbb{P}_{\mathbb{P}^2}(\mathcal{E})$ , where  $\mathcal{E}$  is a rank-2 vector bundle on  $\mathbb{P}^2$ ;
- (iii)  $Z \simeq \mathbb{P}^1$ ,  $\rho(\hat{X}) = 2$ , and  $f$  is a quadric bundle, i.e. any fiber of  $f$  is an irreducible quadric in  $\mathbb{P}^3$ ;
- (iv)  $Z \simeq \mathbb{P}^1 \times \mathbb{P}^1$ ,  $\rho(\hat{X}) = 3$ , and  $f$  is a  $\mathbb{P}^1$ -bundle.

*Proof.* Almost all the statements are proved in [CJR08, Prop. 2.8]. We have to show only that  $Z \not\simeq \mathbb{F}_2$ . Indeed, if  $\dim Z = 2$ , then for a general member  $\bar{S} \in |-\frac{1}{2}K_{\bar{X}}|$ , the restriction  $f|_{\bar{S}} : \bar{S} \rightarrow Z$  is birational. Hence  $Z$  is a del Pezzo surface.  $\square$

**3.9. Theorem.** *Let  $X$  be an imprimitive del Pezzo threefold with at worst terminal Gorenstein singularities. Then there exists a diagram*

$$(3.9.1) \quad \begin{array}{ccc} & \hat{X} & \xrightarrow{\sigma} \bar{X} \\ \xi \swarrow & & \searrow f \\ X & & Z \end{array}$$

where

- (i)  $\xi$  is a  $\mathbb{Q}$ -factorialization;
- (ii)  $\hat{X}$  is an weak del Pezzo threefold with at worst terminal factorial singularities;

- (iii)  $\sigma$  is a blowup in smooth distinct points  $P_1, \dots, P_n \in \bar{X}$ ;
- (iv)  $d(X) = d(\hat{X}) = d(\bar{X}) + n$ ;
- (v)  $\bar{X}$  is a primitive weak del Pezzo threefold with  $\rho(\bar{X}) \leq 2$ , thus  $\bar{X}$  is described by (i)-(iii) of Theorem 3.8.

**3.9.2. Corollary.** *Let  $X$  be a del Pezzo threefold. Then  $r(X) + d(X) \leq 9$ .*

*Proof.* We have  $9 \geq \rho(\bar{X}) + d(\bar{X}) = \rho(\hat{X}) + d(\hat{X}) = r(X) + d(X)$ .  $\square$

**3.9.3. Corollary.** *Let  $X$  be a weak del Pezzo threefold and let  $S \in |-\frac{1}{2}K_X|$  be a smooth element. Then the restriction map  $\text{Cl}(X) \rightarrow \text{Pic}(S)$  is injective and its cokernel is torsion free.*

*Proof.* Clearly the assertion is invariant under taking small modifications. In view of construction (3.9.1), it is sufficient to prove that the restriction map  $\text{Cl}(\bar{X}) \rightarrow \text{Pic}(\bar{S})$  is injective and its cokernel is torsion free, where  $\bar{S} = \sigma(S)$ . Thus we may assume that  $X$  is a primitive factorial weak del Pezzo threefold. The assertion is obvious if  $\rho(X) = 1$ . Assume that  $\rho(X) = 2$ . Then  $\rho(Z) = 1$ . Let  $\Theta$  be the ample generator of  $\text{Pic}(Z)$ . The group  $\text{Cl}(X)$  is generated by  $f^*\Theta$  and the class of  $S$ . Recall that  $Z$  is either  $\mathbb{P}^1$  or  $\mathbb{P}^2$ . Hence  $f^*\Theta|_S$  is either a conic or the pull-back of a line on  $\mathbb{P}^2$ , respectively. It is easy to see that  $f^*\Theta|_S$  and  $-K_S \sim S|_S$  generate a rank 2 primitive sublattice in  $\text{Pic}(S)$ . The case  $\rho(X) = 3$  can be treated similarly.  $\square$

#### 4. PRIMITIVE DEL PEZZO THREEFOLDS WITH $r(X) = 3$

**4.1. Lemma.** *Let  $X$  be a primitive del Pezzo threefold with  $r(X) = 3$  and let  $\mathcal{F} = |F|$  be a complete one-dimensional linear system (pencil) of Weil divisors without fixed components. There is a small  $\mathbb{Q}$ -factorialization  $\xi : \hat{X} \rightarrow X$  such that the proper transform  $\hat{\mathcal{F}}$  of  $\mathcal{F}$  on  $\hat{X}$  is base point free and defines a fibration  $f : \hat{X} \rightarrow \mathbb{P}^1$ . Moreover,  $f$  factors through a (not unique)  $\mathbb{P}^1$ -bundle contraction*

$$(4.1.1) \quad f : \hat{X} \xrightarrow{f_1} \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{f_2} \mathbb{P}^1$$

*Proof.* Take a  $\mathbb{Q}$ -factorialization  $\xi : \hat{X} \rightarrow X$  so that  $\hat{\mathcal{F}}$  is  $\xi$ -nef (one can get it by performing flops over  $X$ ). Then  $\hat{\mathcal{F}}$  is nef. Indeed, otherwise there is a  $K_{\hat{X}}$ -negative extremal ray  $R$  such that  $\hat{\mathcal{F}} \cdot R < 0$ . Since  $\hat{\mathcal{F}}$  has no fixed components,  $R$  must define a flipping contraction. On the other hand,  $K_X$  is Cartier, a contradiction [Mor88, Th. 6.2]. Thus  $\hat{\mathcal{F}}$  is nef. Then  $\hat{\mathcal{F}}$  defines a contraction to a (rational) curve by the base point free theorem. Further, since  $r(X) = 3$ , we have  $\rho(\hat{X}) = 3$ . Running the MMP over  $\mathbb{P}^1$  we obtain  $f_1$ .  $\square$

**4.1.2. Remark-definition.** In notation of (4.1.1), another ruling on  $\mathbb{P}^1 \times \mathbb{P}^1$  defines another pencil  $\mathcal{F}'$  on  $X$ . In this situation, we say that pencils  $\mathcal{F}$  and  $\mathcal{F}'$  are *conjugate*. Thus there is one-to-one correspondence between

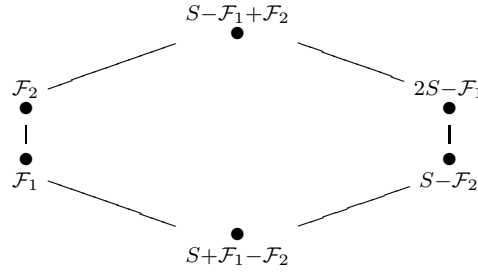
- (i) the set of pairs of conjugate pencils  $\mathcal{F}, \mathcal{F}'$  and
- (ii) the set of  $\mathbb{Q}$ -factorializations  $X' \rightarrow X$  together with a structure of  $\mathbb{P}^1$ -bundle  $f' : \hat{X}' \rightarrow \mathbb{P}^1 \times \mathbb{P}^1$ .

**4.1.3. Corollary.** *The cone of effective divisors  $\overline{\text{NE}}^1(X)$  is generated by classes of pencils  $\mathcal{F}$  as in Lemma 4.1.*

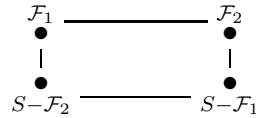
*Proof.* Let  $\xi : \hat{X} \rightarrow X$  be a small  $\mathbb{Q}$ -factorialization. There are natural identifications  $\text{Cl}(X) = \text{Cl}(\hat{X})$  and  $\overline{\text{NE}}^1(X) = \overline{\text{NE}}^1(\hat{X})$ . The variety  $\hat{X}$  is a Mori dream space [HK00]. Hence  $\overline{\text{NE}}^1(\hat{X})$  is a polyhedral cone generated by a finite number of effective divisors  $D_i$ . Running  $D_i$ -MMP on  $\hat{X}$ , after a number of flops, we get a  $\mathbb{P}^1$ -bundle over  $\mathbb{P}^1 \times \mathbb{P}^1$  (because  $X$  is primitive). This shows that  $D_i$  must coincide with some  $\mathcal{F}$ .  $\square$

**4.2. Theorem.** *Let  $X$  be a primitive del Pezzo threefold with  $r(X) = 3$ . Let  $\{\mathcal{F}_i\}$  be the set of all pencils as in Lemma 4.1. Then there are the following possibilities for  $\{\mathcal{F}_i\}$ , where we draw the graph for  $\{\mathcal{F}_i\}$  so that every two elements are connected by an edge if and only if they are conjugate.*

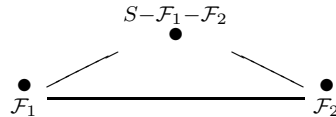
**4.2.1.**  $d(X) = 2$



**4.2.2.**  $d(X) = 4$



**4.2.3.**  $d(X) = 6$  and  $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$



*Proof.* Let  $\mathcal{F}_1$  and  $\mathcal{F}_2$  be two conjugate pencils and let  $\xi : \hat{X} \rightarrow X$  be the corresponding small  $\mathbb{Q}$ -factorialization. Clearly, we have

$$\mathcal{F}_1^2 \equiv \mathcal{F}_2^2 \equiv 0, \quad \mathcal{F}_1 \cdot \mathcal{F}_2 \cdot S = 1, \quad S^2 \cdot \mathcal{F}_1 = S^2 \cdot \mathcal{F}_2 = 2.$$

For any  $j$ , write  $\mathcal{F}_j \sim aS + b_1\mathcal{F}_1 + b_2\mathcal{F}_2$ , where  $a \geq 0$ . Then

$$(4.2.4) \quad \begin{aligned} 0 &= \mathcal{F}_j^2 \cdot S &= a^2d + 4a(b_1 + b_2) + 2b_1b_2, \\ 2 &= \mathcal{F}_j \cdot S^2 &= ad + 2(b_1 + b_2), \end{aligned}$$

where  $d := d(X)$ . Therefore,

$$\begin{aligned} b_1 + b_2 &= \frac{1}{2}(2 - ad), \\ b_1b_2 &= \frac{1}{2}a(ad - 4). \end{aligned}$$

Since this system has an integer solution in  $b_1, b_2$ , the discriminant

$$\frac{1}{4}(2 - ad)^2 - 2a(ad - 4) = \frac{1}{4}(4 - a(8 - d)(ad - 4))$$

must be a square and  $ad$  must be even. Assuming  $a > 0$  (i.e.  $\mathcal{F}_j \neq \mathcal{F}_1, \mathcal{F}_2$ ), we get  $ad = 8, 6, 4$ , or  $2$ . Hence, up to permutation of  $b_1$  and  $b_2$ , there are the following solutions with  $a > 0$ :

$$\begin{aligned} d = 1, & \quad (a, b_1, b_2) = (4, -1, 0), (4, 0, -1); \\ d = 2, & \quad (a, b_1, b_2) = (1, -1, 1), (1, 1, -1), (2, -1, 0), (2, 0, -1); \\ d = 4, & \quad (a, b_1, b_2) = (1, -1, 0), (1, 0, -1); \\ d = 6, & \quad (a, b_1, b_2) = (1, -1, -1). \end{aligned}$$

Note that if  $\mathcal{F}_j$  and  $\mathcal{F}_k$  are conjugate, then  $\mathcal{F}_j \cdot \mathcal{F}_k \cdot S = 1$ . From this one can see that for each  $\mathcal{F}_j$  there are exactly two divisors in  $\{\mathcal{F}_i\}$  that conjugate to  $\mathcal{F}_j$ . Moreover, if  $d \neq 1$ , then conjugacy relations are given by graphs in 4.2.1, 4.2.2, 4.2.3. In the case  $d = 1$  we get the following (disconnected) graph:

$$\begin{array}{ccc} \mathcal{F}_1 & \text{-----} & \mathcal{F}_2 \\ \bullet & & \bullet \end{array} \qquad \begin{array}{ccc} 4S - \mathcal{F}_1 & \text{-----} & 4S - \mathcal{F}_2 \\ \bullet & & \bullet \end{array}$$

Hence there are only two extremal  $K_{\hat{X}}$ -negative contractions on  $\hat{X}$ . On the other hand, the cone  $\overline{\text{NE}}^1(\hat{X})$  has at least three extremal rays, a contradiction.  $\square$

**4.3. Remark.** Let  $X$  be a primitive del Pezzo threefold with  $r(X) = 3$  and  $d(X) = 2$  or  $4$ . Let  $\xi : \hat{X} \rightarrow X$  be a  $\mathbb{Q}$ -factorialization. Then  $\hat{X} \simeq \mathbb{P}(\mathcal{E})$ , where  $\mathcal{E}$  is a stable rank two vector bundle on  $Z = \mathbb{P}^1 \times \mathbb{P}^1$  with  $c_1(\mathcal{E}) = 0$ ,  $c_2(\mathcal{E}) = 6 - d(\mathcal{E})$ .

**4.3.1. Example.** If  $d(X) = 2$ , an example of such  $\mathcal{E}$  can be obtained as a restriction of the null-correlation bundle  $\mathcal{N}$  from  $\mathbb{P}^3$  to  $Z$ , where

$Z \subset \mathbb{P}^3$  is the Segre embedding. Recall that the null-correlation bundle is defined by the exact sequence

$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^3} \longrightarrow \Omega_{\mathbb{P}^3}(2) \longrightarrow \mathcal{N}(1) \longrightarrow 0.$$

Its projectivization  $Y := \mathbb{P}(\mathcal{N})$  is a Fano fourfold of index 2 [SW90]. This  $Y$  has also a structure of  $\mathbb{P}^1$ -bundle over a smooth three-dimensional quadric. Let  $\hat{X} = \mathbb{P}(\mathcal{E}) = \pi^{-1}(Z)$ , where  $\pi : Y \rightarrow \mathbb{P}^3$  is the natural projection. Then  $\hat{X}$  is a weak del Pezzo threefold of type 4.2.2.

Examples of del Pezzo threefolds of type 4.2.1 can be constructed similarly by restricting to  $Z \subset \mathbb{P}^3$  rank two stable vector bundles  $\mathcal{F}$  with  $c_1 = 0$ ,  $c_2 = 2$  [Har78, §9].

Another way to show existence of del Pezzo threefolds of types 4.2.2 and 4.2.1 is by writing down explicit equations:

**4.3.2. Example.** Let  $X \subset \mathbb{P}^5$  is given by the equations

$$\begin{cases} x_1x_3 - x_2x_4 + a_{3,4}x_3x_5 + a_{3,6}x_3x_6 + a_{4,5}x_4x_5 + a_{4,6}x_4x_6 = 0 \\ x_1x_5 - x_2x_6 + b_{3,4}x_3x_5 + b_{3,6}x_3x_6 + b_{4,5}x_4x_5 + b_{4,6}x_4x_6 = 0 \end{cases}$$

where  $a_{i,j}$ ,  $b_{i,j}$  are sufficiently general constants. Then  $X$  is a del Pezzo threefold having exactly 4 nodes. By Corollary 10.6.2  $r(X) \geq 3$ . On the other hand, by results of 7.5 and §8 below  $r(X) = 3$ . Finally, two quadrics  $x_5 = x_6 = x_1x_3 - x_2x_4 = 0$  and  $x_3 = x_4 = x_1x_5 - x_2x_6 = 0$  determine two conjugate pencils. Therefore,  $X$  is of type 4.2.2.

## 5. DEL PEZZO THREEFOLDS WITH $r(X) = 2$

The results of this section are contained in [JP08]. We give a short self-contained proof for the convenience of the reader.

**5.1.** Let  $X$  be a del Pezzo threefold with  $r(X) = 2$ . There exists the following diagram:

$$\begin{array}{ccccc} & \hat{X} & \dashrightarrow & \hat{X}^+ & \\ f \swarrow & & \searrow \xi & \swarrow \xi^+ & \searrow f^+ \\ Z & & X & & Z^+ \end{array}$$

where  $\xi, \xi^+$  are small  $\mathbb{Q}$ -factorializations,  $\hat{X} \dashrightarrow \hat{X}^+$  is a flop, and  $f, f^+$  are  $K$ -negative extremal contractions. We may assume that  $\dim Z \geq \dim Z^+$ . Let  $S = -\frac{1}{2}K_X$  and let  $\hat{S} = h^*S$ . Let  $M$  (resp.  $M^+$ ) be the ample generator of  $\text{Pic}(X)$  (resp.  $\text{Pic}(X^+)$ ). Put  $L := f^*M$  and  $L^+ := f^{+*}M^+$ . Let  $L'$  be the proper transform of  $L^+$  on  $\hat{X}$ . If  $f$  is birational, then  $E \subset \hat{X}$  denotes the  $f$ -exceptional divisor. Similarly, if  $f^+$  is birational, then  $E' \subset \hat{X}$  is the proper transform of  $f^+$ -exceptional

divisor. Only one such a solution has  $a = 0$ . Hence the case  $d = 1$  is impossible and for  $d = 6$  we have  $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ .

**5.1.1. Remark.** If in the above notation  $d(X) \leq 2$ , then by 3.4.1 there is a natural (Bertini or Geiser) involution  $\tau : X \rightarrow X$ . In this case, we can take  $\hat{X}^+ \simeq \hat{X}$  and  $\xi^+ = \tau \circ \xi$ . Therefore,  $Z \simeq Z^+$  and  $f^+$  has the same type as  $f$ .

The following theorem was proved (in much stronger form) in [JP08]. For convenience of the reader we provide a short proof.

**5.2. Theorem.** *In the above notation there are the following possibilities.*

|          | $f$                    | $f^+$                  | d | $\text{Pic}(\hat{X})$  | s                   |
|----------|------------------------|------------------------|---|------------------------|---------------------|
| (5.2.1)  | $\mathbb{P}^1$ -bundle | $\mathbb{P}^1$ -bundle | 1 | $L + L' \sim 6\hat{S}$ | 22                  |
| (5.2.2)  |                        |                        | 2 | $L + L' \sim 3\hat{S}$ | 11                  |
| (5.2.3)  |                        |                        | 3 | $L + L' \sim 2\hat{S}$ | 6                   |
| (5.2.4)  |                        |                        | 6 | $L + L' \sim \hat{S}$  | 0                   |
| (5.2.5)  | $\mathbb{P}^1$ -bundle | quadric bundle         | 5 | $L + L' \sim \hat{S}$  | 1                   |
| (5.2.6)  | quadric bundle         | quadric bundle         | 1 | $L + L' \sim 4\hat{S}$ | $\leq 22$           |
| (5.2.7)  |                        |                        | 2 | $L + L' \sim 2\hat{S}$ | $\leq 11$           |
| (5.2.8)  |                        |                        | 4 | $L + L' \sim \hat{S}$  | $\leq 3$            |
| (5.2.9)  | birational             | $\mathbb{P}^1$ -bundle | 4 | $E + L' \sim \hat{S}$  | 3                   |
| (5.2.10) |                        |                        | 7 | $E + 2L' \sim \hat{S}$ | 0                   |
| (5.2.11) | birational             | quadric bundle         | 3 | $E + L' \sim \hat{S}$  | 4, 5, 6             |
| (5.2.12) | birational             | birational             | 1 | $E + E' \sim 2\hat{S}$ | $12 \leq s \leq 22$ |
| (5.2.13) |                        |                        | 2 | $E + E' \sim \hat{S}$  | $6 \leq s \leq 11$  |

Here in the 5th column we indicate relations between  $L$ ,  $L'$ ,  $E$ , and  $E'$  in  $\text{Pic}(\hat{X})$ .

*Proof.* First we consider the case where  $X$  is primitive, i.e. both  $f$  and  $f^+$  are of fiber type. Write  $L' \sim a\hat{S} + bL$ . Clearly,  $a > 0$ . Since  $L'$  is not ample,  $b \leq 0$ . Since  $L'$  and  $\hat{S}$  generate  $\text{Pic}(\hat{X})$ , we have  $b = -1$ .

Let  $n := \dim Z$  and  $n' := \dim Z^+$ . Further,

$$\hat{S}^3 = d, \quad \hat{S}^2 \cdot L = n + 1, \quad \hat{S} \cdot L^2 = n - 1, \quad L^3 = 0$$

and similarly

$$\hat{S}^2 \cdot L' = n' + 1, \quad \hat{S} \cdot L'^2 = n' - 1.$$

This gives us

$$n' + 1 = \hat{S}^2 \cdot L' = ad - (n + 1), \quad ad = n + n' + 2.$$

On the other hand, by Remark 5.1.1  $d \geq 3$  whenever  $n \neq n'$ . This gives us the possibilities (5.2.1) – (5.2.8) in our table.

Assume that  $f$  is birational. If  $Z \simeq \mathbb{P}^3$ , then we get the case 5.2.10. Thus we may assume that  $E$  and  $S \sim L - E$  generate  $\text{Pic}(\hat{X})$ . Assume that  $f^+$  is of fiber type. As above,  $L' \sim a\hat{S} - E$  and  $n' + 1 = \hat{S}^2 \cdot L' = ad - 1$ . So,  $ad = n' + 2 \leq 4$ . On the other hand, by Remark 5.1.1  $d \geq 3$ . Hence  $a = 1$  and  $d = n' + 2$ . This gives us (5.2.9) and (5.2.11).

Finally assume that both  $f$  and  $f^+$  are birational. Since  $\text{Pic}(\hat{X}) = \mathbb{Z} \cdot \hat{S} \oplus \mathbb{Z} \cdot E = \mathbb{Z} \cdot \hat{S} \oplus \mathbb{Z} \cdot E'$  and  $\dim |E'| = 0$ , we can write  $E' \sim a\hat{S} - E$ . Hence,

$$1 = E' \cdot \hat{S}^2 = (a\hat{S} - E) \cdot \hat{S}^2 = ad - 1, \quad ad = 2.$$

We get cases (5.2.12) and (5.2.13).  $\square$

**5.3. Corollary.** *Let  $X$  be a del Pezzo threefold with  $r(X) = 1$ . Assume that  $X$  is singular. Then  $d(X) \leq 4$ . If  $d(X) = 4$ , then every singular point  $P \in X$  is rs-nondegenerate (see 10.1). Moreover,  $\lambda(X, P) = \nu(X, P)$  and  $\sum_P \lambda(X, P) \leq 2$ .*

*Proof.* Let  $P \in X$  be a general point. Let  $\sigma : \tilde{X} \rightarrow X$  be the blowup of  $P$ , let  $E := \sigma^{-1}(P)$ , and let  $\tilde{S}$  be the proper transform of  $S = -\frac{1}{2}K_X$ . Write  $-K_{\tilde{X}} = 2\sigma^*S - 2E = 2\tilde{S}$ . Since the linear system  $|\tilde{S}|$  is base point free and big,  $\tilde{X}$  is a weak del Pezzo threefold with at worst factorial terminal singularities,  $\rho(\tilde{X}) = 2$ , and  $d(\tilde{X}) = d(X) - 1$ . If  $d(X) \geq 5$ , then by Theorem 5.2 we have only one possibility 5.2.9. But then both  $\tilde{X}$  and  $X$  are smooth. If  $d(X) = 4$ , then we have case 5.2.11. In this case any singularity  $\tilde{P} \in \tilde{X}$  is analytically isomorphic to the hypersurface singularity given by  $x_1x_2 + x_3^2 + x_4^n = 0$ . Then  $\lambda(\tilde{X}, \tilde{P}) = \nu(\tilde{X}, \tilde{P}) = \lfloor n/2 \rfloor$ . The last inequality follows by Proposition 10.6  $\square$

**5.4.** By [JP08] all the cases in the table do occur<sup>†</sup>. Below we give explicit examples of some del Pezzo threefolds with  $r(X) = 2$ .

<sup>†</sup>There is a typographical error in [JP08, Th. 3.6]: the case  $c_2(\mathcal{F}) = 6$  occurs.



**5.4.1. Case 5.2.3.**  $X = X_3 \subset \mathbb{P}^4$  is given by an equation of the form

$$(x_1x_4 - x_2x_3)\ell_1 + (x_2^2 - x_1x_3)\ell_2 + (x_3^2 - x_1x_3)\ell_3 = 0,$$

where  $\ell_i(x_1, \dots, x_5)$  are linear forms.

**5.4.2. Case 5.2.5** (cf. 7.4 and 8.3.) Let  $Y$  be the blowup of  $\mathbb{P}^1 \times \mathbb{P}^2$  along a smooth curve  $C$  of bidegree  $(2, 1)$ . Then  $Y$  is a Fano threefold with  $-K_Y^3 = 38$  and  $\rho(Y) = 3$  [MM82]. Let  $S \subset \mathbb{P}^1 \times \mathbb{P}^2$  be a (unique) effective divisor of bidegree  $(0, 1)$  containing  $C$  and let  $\tilde{S}$  be the proper transform of  $S$  on  $Y$ . Then  $\tilde{S} \simeq S \simeq \mathbb{P}^1 \times \mathbb{P}^1$  and  $\mathcal{O}_{\tilde{S}}(\tilde{S})$  is of type  $(-1, -1)$ . Therefore, there exists a contraction  $\varphi : Y \rightarrow X$ , where  $\varphi(\tilde{S})$  is a node. Here  $X$  is a quintic del Pezzo threefold as in 5.2.5.

**5.4.3. Case 5.2.7.**  $X \subset \mathbb{P}(1^4, 2)$  is given by the equation

$$x_5^2 = (x_1x_2 - x_3x_4)^2 + (x_1x_2 - x_3x_4)q_1(x_1, \dots, x_4) + q_2(x_1, \dots, x_4)^2,$$

where  $q_1$  and  $q_2$  are general quadratic forms.

**5.4.4. Case 5.2.8.**  $X \subset \mathbb{P}^5$  is given by the equations

$$x_1x_2 + x_3x_4 + x_5^2 + x_6l_1(x_1, \dots, x_6) = x_1x_3 + x_6l_2(x_1, \dots, x_6) = 0,$$

where  $l_i$  are linear forms. It is easy to see that  $X$  contains two singular quadrics  $Q_1 = \{x_6 = x_1 = x_3x_4 + x_5^2 = 0\}$  and  $Q_2 = \{x_6 = x_3 = x_1x_2 + x_5^2 = 0\}$ . They generate two pencils. Hence  $X$  is of type 5.2.8. For a general choice of  $l_i$  the variety  $X$  has exactly one node.

**5.4.5. Case 5.2.9.**  $X \subset \mathbb{P}^5$  is given by the equations

$$x_3x_4 - x_5^2 + x_6l_1(x_1, \dots, x_6) = x_1x_4 - x_2x_5 + x_6l_2(x_1, \dots, x_6) = 0,$$

where  $l_i$  are general linear forms. Its singular locus consists of three points

$$\{x_3 = x_4 = x_5 = x_6 = l_1 = 0\}, \{x_2 = x_4 = x_5 = x_6 = x_3l_2 - x_1l_1 = 0\}$$

and  $X$  contains the plane  $\{x_4 = x_5 = x_6 = 0\}$ .

**5.4.6. Case 5.2.11.** Let  $X \subset \mathbb{P}^4$  be given by the following equation:

$$x_1u(x_1, x_2, x_3, x_4, x_5) + x_2v(x_1, x_2, x_3, x_4, x_5) = 0,$$

where  $u$  and  $v$  are quadratic forms. This cubic contains the plane  $\Pi := \{x_1 = x_2 = 0\}$  and, for general  $u$  and  $v$ , the singular locus consists of four nodes. The projection from  $\Pi$  gives us a quadric bundle structure on  $\hat{X}$  (which is the blowup of  $\Pi$ ). For some special choices of  $u$  and  $v$  the cubic  $X$  can have one or two extra (factorial) singular points (see [FW]) and  $r(X) = 2$ .

**5.4.7. Case 5.2.12.**  $X \subset \mathbb{P}(1^3, 2, 3)$  is given by the equation

$$x_5^2 = x_4^3 + x_4^2\phi_2 + x_4\phi_4 + \phi_3^2,$$

where  $\phi_i(x_1, x_2, x_3)$  are general homogeneous forms of degree  $i$ .

**5.4.8. Case 5.2.13.**  $X \subset \mathbb{P}(1^4, 2)$  is given by the equation

$$x_5^2 = x_1\phi_3(x_1, \dots, x_4) + q(x_1, \dots, x_4)^2,$$

where  $\phi_3$  and  $q$  are general homogeneous forms of degree 3 and 2, respectively.

## 6. ROOT SYSTEMS

**6.1.** Let  $X$  be a del Pezzo threefold of degree  $d = d(X)$ . In this section we study the image of the restriction map  $\iota^* : \text{Cl}(X) \rightarrow \text{Pic}(S)$ , where  $S \in |-\frac{1}{2}K_X|$  is a smooth member contained in the smooth locus of  $X$  and  $\iota : S \hookrightarrow X$  is an embedding. Define  $\Delta$  and  $\Delta'$  as in 1.5. If  $X$  is imprimitive, we apply construction (3.9.1) with all corresponding notation. In the primitive case, to unify notation, we put  $\sigma = \text{id}$ .

Note that  $S$  does not pass through singular points of  $X$ . Thus we may identify  $S$  and  $\hat{S} = \xi^{-1}(S)$ . Let  $\bar{S} := \sigma(S)$ . Then  $\bar{S}$  is a smooth del Pezzo surface,  $\bar{S} \in |-\frac{1}{2}K_{\bar{X}}|$  and  $\sigma_S : S \rightarrow \bar{S}$  is a blowup of  $r(X) - r(\bar{X})$  distinct points. Define  $\bar{\Delta}$  and  $\bar{\Delta}'$  for  $\bar{S}$  as in 1.5.

**6.2. Theorem.** (i) *In the above notation the image  $\iota^* \text{Cl}(X)$  is the orthogonal complement to  $\Delta'$ . In particular,*

$$(6.2.1) \quad \text{rk } \Delta' + \text{rk } \text{Cl}(X) + d(X) = 10.$$

(ii) *We have  $\Delta' = \sigma_S^* \bar{\Delta}'$ .*

(iii) *According to possibilities for  $Z$  we have the following cases:*

- (a) *If  $Z$  is a point (i.e.  $\rho(\bar{X}) = 1$ ), then  $\bar{\Delta}' = \bar{\Delta}$ . Here  $\bar{\Delta}'$  is of type  $E_8, E_7, E_6, D_5, A_4, A_1$  in cases  $d(\bar{X}) = 1, 2, 3, 4, 5$ , and 8, respectively.*
- (b) *If  $Z \simeq \mathbb{P}^2$ , then  $\bar{\Delta}' = \{\alpha \in \bar{\Delta} \mid \alpha \cdot f^*K_Z = 0\}$ . Here  $\bar{\Delta}'$  is of type  $A_m$  (recall that  $d(\bar{X}) = 1, 2, 3, 5$ , or 6).*
- (c) *If  $Z \simeq \mathbb{P}^1$ , then  $\bar{\Delta}' = \{\alpha \in \bar{\Delta} \mid \alpha \cdot C = 0\}$ , where  $C$  is a conic on  $\bar{S}$ . Here  $\bar{\Delta}'$  is of type  $D_m$  (recall that  $d(\bar{X}) = 1, 2$ , or 4).<sup>‡</sup>*
- (d) *If  $X \simeq (\mathbb{P}^1)^3$ , then  $\Delta'$  is the subsystem  $A_1$  in  $\Delta \simeq A_1 \times A_2$ .*
- (e) *If  $X$  is of type 4.2.1 or 4.2.2, then  $\Delta'$  is of type  $A_5$  or  $A_3$ , respectively.*

*Proof.*

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<sup>‡</sup>Cases (b) and (c) overlap for  $X$  with  $d(X) = 5$ .

**6.3.** Assume that  $X$  is primitive. Then  $\hat{X} = \bar{X}$  and  $\sigma = \text{id}$ . All the statements are obvious if  $r(X) = 1$ . We assume that  $r(X) \geq 2$ . Let  $f : \hat{X} \rightarrow Z$  be an extremal  $K_{\hat{X}}$ -negative contraction. Let  $S \in |-\frac{1}{2}K_{\hat{X}}|$  be a smooth member. Denote by  $\delta : S \rightarrow Z$  the restriction of  $f$  to  $S$ . Since  $f : \hat{X} \rightarrow Z$  is an extremal contraction, the image of  $\iota^* : \text{Pic}(\hat{X}) \rightarrow \text{Pic}(S)$  is generated by  $\delta^* \text{Pic}(Z)$  and  $-K_S = -\frac{1}{2}K_{\hat{X}}|_S$ . Clearly,  $f : S \rightarrow Z$  is surjective. Fix a standard basis in  $\text{Pic}(S)$  [Dol, ch. 8]:

$$\mathbf{h}, \quad \mathbf{e}_1, \dots, \mathbf{e}_n,$$

where  $n = 9 - d$  and

$$\mathbf{h}^2 = 1, \quad \mathbf{e}_i^2 = -1, \quad \mathbf{e}_i \cdot \mathbf{e}_j = 0 \quad \text{for } i \neq j.$$

Since  $\iota^* \text{Pic}(\hat{X})$  is generated by  $\delta^* \text{Pic}(Z)$  and  $-K_S$ , we have

$$\Delta' = \{\alpha \in \Delta \mid \alpha \cdot \delta^* \text{Pic}(Z) = 0\}.$$

**6.3.1. Case  $Z \simeq \mathbb{P}^2$  and  $f$  is a  $\mathbb{P}^1$ -bundle.** Then  $f : S \rightarrow \mathbb{P}^2$  is the blowup of  $n = 9 - d$  points and we can choose the basis  $\mathbf{h}, \mathbf{e}_1, \dots, \mathbf{e}_n$  so that  $\mathbf{h} = f^* \mathcal{O}_{\mathbb{P}^2}(1)$  and  $\mathbf{e}_1, \dots, \mathbf{e}_n$  are  $f$ -exceptional. In this case,  $\iota^* \text{Pic}(\hat{X})$  is generated by  $\mathbf{h}$  and  $K_S$ . Hence  $\Delta' = \{\alpha \in \Delta \mid \alpha \cdot \mathbf{h} = 0\}$ . Then  $\Delta' = \{\mathbf{e}_i - \mathbf{e}_j \mid i \neq j\}$  is a root subsystem of rank  $n - 1$  generated by  $\mathbf{e}_1 - \mathbf{e}_2, \dots, \mathbf{e}_{n-1} - \mathbf{e}_n$ . Thus  $\Delta'$  is of type  $A_{n-1}$ .

**6.3.2. Case  $Z \simeq \mathbb{P}^1$ , i.e.  $f$  is a quadric bundle.** Then  $n \geq 4$  and  $\delta : S \rightarrow \mathbb{P}^1$  is a conic bundle. Let  $C$  be a fiber. By changing the basis  $\mathbf{h}, \mathbf{e}_1, \dots, \mathbf{e}_n$  we may assume that  $C \sim \mathbf{h} - \mathbf{e}_1$ . Then  $\Delta' = \{\alpha \in \Delta \mid \alpha \cdot C = 0\}$ , i.e.  $\Delta'$  consists of the following elements:

- $\mathbf{e}_i - \mathbf{e}_j, \quad i, j > 1, i \neq j.$
- $\pm(\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_i - \mathbf{e}_j), \quad i, j > 1, i \neq j.$

Simple roots can be taken as follows:

$$\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad \mathbf{e}_2 - \mathbf{e}_3, \dots, \mathbf{e}_{n-1} - \mathbf{e}_n.$$

Hence  $\Delta'$  is of type  $D_{n-1}$  if  $n \geq 5$  and  $A_3$  if  $n = 4$ .

**6.3.3. Case  $Z \simeq \mathbb{P}^1 \times \mathbb{P}^1$  and  $f$  is a  $\mathbb{P}^1$ -bundle.** Let  $\ell_i := F_i|_S$ . Then we may assume that  $\ell_1 \sim \mathbf{h} - \mathbf{e}_1, \ell_2 \sim \mathbf{h} - \mathbf{e}_2$ .  $\Delta'$  consists of the following elements:

- $\mathbf{e}_i - \mathbf{e}_j, \quad i, j > 2, i \neq j.$
- $\pm(\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_i), \quad i > 2.$

Simple roots can be taken as follows:

$$\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad \mathbf{e}_3 - \mathbf{e}_4, \dots, \mathbf{e}_{n-1} - \mathbf{e}_n.$$

Thus  $\Delta'$  is of type  $A_{n-2}$ .

This proves our theorem in the case where  $X$  is primitive.

**6.4.** Now consider the case where  $X$  is imprimitive. Obviously, the statement of (iii) follows from 6.3. There is a birational contraction  $\sigma : \hat{X} \rightarrow \bar{X}$ , where  $\bar{X}$  is primitive and  $\sigma$  is a composition of blowups of smooth points. Let  $l := r(X) - r(\bar{X})$ , let  $E_1, \dots, E_l$  be  $\sigma$ -exceptional divisors, and let  $\mathbf{e}_i = E_i \cap S$  for  $i = 1, \dots, l$ . By the above, the statement of our theorem holds for  $\bar{X}$  with root system  $\bar{\Delta}' \subset \bar{\Delta} \subset \text{Pic}(\bar{S})$ . We have a commutative diagram

$$\begin{array}{ccc} \text{Pic}(\bar{S}) \xrightarrow{\sigma_S^*} \text{Pic}(S) & \simeq & \text{Pic}(\bar{S}) \oplus \sum_{i=1}^l \mathbf{e}_i \cdot \mathbb{Z} \\ \uparrow \iota^* & & \uparrow \iota^* \quad \parallel \\ \text{Cl}(\bar{X}) \xrightarrow{\sigma^*} \text{Cl}(X) & \simeq & \text{Cl}(\bar{X}) \oplus \sum_{i=1}^l E_i \cdot \mathbb{Z} \end{array}$$

Now it is easy to see that  $\iota^* \text{Cl}(X)^\perp \subset \sigma_S^* \text{Pic}(\bar{S})$ . Therefore,

$$\sigma_S^* \bar{\Delta}' \subset \Delta \cap \iota^* \text{Cl}(X)^\perp \subset \Delta \cap \sigma_S^* \text{Pic}(\bar{S}).$$

On the other hand,  $\sigma_S^* \bar{\Delta}' \supset \Delta \cap \sigma_S^* \text{Pic}(\bar{S})$ . Hence,  $\sigma_S^* \bar{\Delta}' = \Delta \cap \iota^* \text{Cl}(X)^\perp$ . This proves (ii). As a consequence we have that the left hand side of (6.2.1) is preserved under birational contractions  $\sigma$ . By 6.3 the equality (6.2.1) holds for primitive del Pezzo threefolds. Thus (6.2.1) holds for imprimitive ones as well. This proves (i). □

## 7. DEL PEZZO THREEFOLDS WITH MAXIMAL $r(X)$

Recall that  $r(X) + d(X) \leq 9$  by Corollary 3.9.2. In this section we study del Pezzo threefolds with  $r(X) + d(X) = 9$ .

We say that points  $P_1, \dots, P_n \in \mathbb{P}^3$  are *in general position* if no three of them lie on one line and no four of them lie on one plane.

**7.1. Theorem.** *Let  $X$  be a del Pezzo threefold with  $r(X) + d(X) = 9$ . Assume that  $X \not\cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . Then*

- (i)  *$X$  can be obtained by applying construction (3.9.1) to  $\mathbb{P}^3 \simeq V_8 \subset \mathbb{P}^9$  where  $\sigma$  is the blowup of  $n := r(X) - 1$  points  $P_1, \dots, P_n \in V_8$  in general position.*
- (ii) *Singular points of  $X$  are images of proper transforms of*
  - (a) *lines passing through  $P_i$  and  $P_j$ ,  $i \neq j$ ,*
  - (b) *twisted cubics passing through six distinct points  $P_{i_1}, \dots, P_{i_6}$  (see Claim 7.1.2 below).*
- (iii) *If all the singularities of  $X$  are nodes, then  $s(X) = 28, 16, 10, 6, 3, 1$  in cases  $d(X) = 1, 2, 3, 4, 5, 6$ , respectively.*
- (iv) *If  $d(X) \geq 2$ , then all the singularities of  $X$  are nodes.*

Conversely, assume that  $X$  is a del Pezzo threefold whose singularities are at worst nodes and assume that  $s(X) = 28, 16, 10, 6, 3, 1$  in cases  $d(X) = 1, 2, 3, 4, 5, 6$ , respectively. Then  $d(X) + r(X) = 9$ .

Note that in the case  $d(X) = 1$  the statement of (iv) is wrong: one can easily construct  $X$  having only 27 singular points, where one of them is not a node.

**7.1.1. Corollary.** *Let  $X$  be a del Pezzo threefold with  $r(X) + d(X) = 9$ . If  $d(X) \geq 3$  and  $d(X) \neq 6$ , then  $X$  is unique up to isomorphism. If  $d(X) = 2$  (resp.  $d(X) = 1$ ), then  $X$  belongs to a 3-dimensional (resp. 6-dimensional) family. There are exactly two isomorphism classes of Pezzo threefolds with  $d(X) = 6$ ,  $r(X) = 3$ .*

*Proof.* (i) If  $X$  is primitive, then either  $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$  or  $X \simeq \mathbb{P}^3$  by Theorems 3.2, 4.2, and 5.2. Thus we assume that  $X$  is imprimitive and  $d(X) \leq 7$ . We use notation of Theorem 3.9. Run construction (3.9.1) in such a way that  $n$  is maximal possible. On the last step we get a primitive weak del Pezzo threefold  $\bar{X}$  with  $\rho(\bar{X}) = 9 - d(\bar{X})$ . Moreover, if  $\rho(\bar{X}) = 3$ , then  $n = 0$ ,  $\rho(\hat{X}) = r(X) = 3$ , and  $d(X) = 6$ . By Theorem 4.2 we have  $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . On the other hand, by Theorem 5.2  $\rho(\bar{X}) \neq 2$ . Hence  $\rho(\bar{X}) = 1$ ,  $d(\bar{X}) = 8$ , and then  $\bar{X} \simeq \mathbb{P}^3$ .

It remains to show that the centers  $P_1, \dots, P_n$  of the blowup  $\hat{X} \rightarrow \bar{X} \simeq \mathbb{P}^3$  are in general position. Indeed, if distinct points  $P_i, P_j, P_k$  lie on a line  $L \subset \mathbb{P}^3$ , then for its proper transform  $L'$  on  $\hat{X}$  we have  $-K_{\hat{X}} \cdot L' = -K_{\mathbb{P}^3} \cdot L - 3 \cdot 2 < 0$ , a contradiction. Similarly, if four distinct points  $P_i, P_j, P_k, P_l$  lie on a plane  $D \subset \mathbb{P}^3$ , then for its proper transform  $\hat{D}$  on  $\hat{X}$  we have  $K_{\hat{X}}^2 \cdot \hat{D} = K_{\mathbb{P}^3}^2 \cdot D - 4 \cdot 4 = 0$ . Hence  $\hat{D}$  is contracted by the anticanonical map, a contradiction. This proves (i).

(ii) Let  $P \in X$  be a singular point. Then  $\xi^{-1}(P)$  is a curve. Let  $\hat{C} \subset \xi^{-1}(P)$  be a component and let  $\bar{C} := \sigma(\hat{C}) \subset \bar{X}$ . There are two members  $\hat{S}', \hat{S}'' \in |-\frac{1}{2}K_{\hat{X}_0}|$  such that  $C \subsetneq \hat{S}' \cap \hat{S}''$ . Then  $\bar{C} \subsetneq \bar{S}' \cap \bar{S}''$ , where  $\bar{S}', \bar{S}'' \subset \hat{X}_0 = \mathbb{P}^3$  are proper transforms of  $\hat{S}'$  and  $\hat{S}''$ . Therefore,  $\deg \bar{C} \leq 3$  and  $\bar{C}$  is not a plane cubic. If  $\deg \bar{C} = 2$ , then  $\bar{C}$  is a conic and it must contain four distinct points from  $P_1, \dots, P_n$ . This contradicts our assumption that  $P_1, \dots, P_n$  are in general position. Therefore,  $\bar{C}$  is either a line or a twisted cubic. This proves (ii).

(iii) follows by Corollary 10.6.2.

(iv) If  $d(X) \geq 3$ , then  $X$  is unique up to isomorphism and the statement (iv) can be checked directly (see 7.3-7.6 below). Let  $d(X) = 2$  the  $\xi$ -exceptional set consists of proper transforms of lines  $L_{i,j}$  passing through pairs of distinct points  $P_i, P_j$  and one twisted cubic  $C$  passing through  $P_1, \dots, P_6$ . Moreover, the lines  $L_{i,j}$  meet  $C$  transversely. By

blowing the points  $P_1, \dots, P_6$  up we get these curves disjointed. Thus  $\xi$  is a small resolution whose exceptional set is a disjointed union of 16 smooth rational curves.

The last assertion follows by Corollary 10.6.2.  $\square$

**7.1.2. Claim.** *Let  $P_1, \dots, P_6 \in \mathbb{P}^3$  be a points in general position. Then there exists a twisted cubic curve  $C = C_3 \subset \mathbb{P}^3$  containing  $P_1, \dots, P_6$ . Such a curve is unique.*

*Proof.* It is easy and left to the reader.  $\square$

By Theorem 6.2 we have the following.

**7.1.3. Corollary.** *Let  $X$  be a del Pezzo threefold with  $r(X) = 9 - d(X)$  and  $d(X) \leq 5$ . Then the image of  $\iota^* : \text{Cl}(X) \rightarrow \text{Pic}(S)$  is a sublattice orthogonal to some root  $\alpha \in \Delta$ , i.e.  $\Delta' = \{\pm\alpha\}$ . Moreover,  $\Delta''$  is of type  $E_7, D_6, A_5, A_1 \times A_3, A_2$  in cases  $d(X) = 1, 2, 3, 4, 5$ , respectively.*

**7.1.4. Corollary.** *Let  $X$  be a del Pezzo threefold with  $r(X) = 9 - d(X)$  and  $d(X) \leq 4$ .*

- (i) *If  $d(X) \neq 2$ , then the image of the natural map  $G \rightarrow \text{Aut}(\Delta'')$  is contained in the Weyl group  $W(\Delta'')$ .*
- (ii) *If  $d(X) \leq 3$  and  $\mathbb{k}$  is algebraically closed (i.e. we are in the geometric case), then the map  $G \rightarrow \text{Aut}(\Delta'')$  is an embedding.*

*Proof.* (i) Similar to [Man67, Ch. 4, 26.5]. If  $d(X) = 1$ , then  $\Delta''$  is of type  $E_7$  and  $\text{Aut}(\Delta'') = W(\Delta'')$  [Ser87]. For  $d(X) = 3$  and 4 the group  $\text{Aut}(\Delta'')$  is a direct product of  $W(\Delta'')$  and  $\pm \text{id}$ . If the image of  $G$  is not contained in  $W(\Delta'')$ , then the element  $\tau := -\text{id}$  can be expressed as  $gw$ , where  $g \in G$  and  $w \in W(\Delta'')$ . Note that any reflection  $s \in W(\Delta'')$  can be extended to an element  $\text{Aut}(\iota^* \text{Cl}(X))$ . Hence, the action of  $\tau$  can be extended to an action on  $\iota^* \text{Cl}(X)$  so that  $\tau(K_S) = gw(K_S) = K_S$ . Let  $E$  be a plane on  $X$  and let  $\mathbf{e}$  be the class  $\iota^*(E)$ . Then

$$\tau(\mathbf{e}) = \tau\left(\frac{1}{d}K_S + \mathbf{e}\right) - \frac{1}{d}\tau(K_S) = -\left(\frac{1}{d}K_S + \mathbf{e}\right) - \frac{1}{d}K_S = -\frac{2}{d}K_S - \mathbf{e}.$$

In particular,  $2/d$  must be integral, a contradiction.

(ii) Let  $G_0$  be the kernel of the map  $G \rightarrow \text{Aut}(\Delta'')$ . Then  $G_0$  acts trivially on  $\text{Cl}(X)$ . In particular, the diagram (3.9.1) is  $G_0$ -equivariant. Thus  $G_0$  acts on  $\bar{X} = \mathbb{P}^3$  so that there are  $\geq 5$  fixed points in general position, the images of  $\sigma$ -exceptional divisors. Then  $G_0$  must be trivial.  $\square$

**7.2. Theorem.** *Let  $X$  be a del Pezzo threefold with  $r(X) + d(X) = 9$ . Assume that  $X \not\cong \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . Let  $\Pi \subset X$  is a plane, let  $\hat{\Pi} \subset \hat{X}$  be its proper transform, and let  $\bar{\Pi} = \sigma(\hat{\Pi}) \subset \bar{X} = \mathbb{P}^3$ . Then  $\bar{\Pi}$  is of one of the following types:*

- (i)  $\bar{\Pi}$  is one of the points  $P_i$ ,  $\hat{\Pi}$  is  $\sigma$ -exceptional;

- (ii)  $\bar{\Pi}$  is a plane passing through three of the points  $P_i$ ;
- (iii)  $\bar{\Pi}$  is quadratic cone passing through six of the points  $P_i$  so that one of them is the vertex of the cone;
- (iv) (only for  $d(X) = 1$ )  $\bar{\Pi}$  is cubic surface passing through all the points  $P_i$  so that four of them are double points;
- (v) (only for  $d(X) = 1$ )  $\bar{\Pi}$  is quartic surface passing through all the points  $P_i$  so that all of them are double points and one of them is a triple point.

The number of planes on  $X$  is given by the following table:

|        |   |   |   |   |    |    |     |
|--------|---|---|---|---|----|----|-----|
| $d(X)$ | 7 | 6 | 5 | 4 | 3  | 2  | 1   |
| $p(X)$ | 1 | 2 | 4 | 8 | 15 | 32 | 126 |

*Proof.* It is easy to see that all the subvarieties  $\Pi$  described in (i)-(v) are planes. So the number of planes is at least the number indicated in the table. On the other hand, for any plane  $\Pi \subset X$ , the intersection  $\Pi \cap S$  is a line whose class in  $\text{Pic}(S)$  is orthogonal to the root  $\alpha \in \text{Pic}(S)$  (see Corollary 7.1.3). Define

$$\mathcal{E} := \{e \in \text{Pic}(S) \mid e^2 = K_S \cdot e = -1, e \cdot \Delta' = 0\}.$$

Thus the number of planes is at most  $|\mathcal{E}|$ .

Let  $\mathbf{h}, \mathbf{e}_1, \dots, \mathbf{e}_{9-d}$  be a standard basis of  $\text{Pic}(S)$ . Since cases  $n \leq 3$  are trivial, we may assume that  $n \geq 4$ . Then the Weil group  $W(\Delta)$  transitively acts on  $\Delta$  [Dol, 8.2.14] and we can take it so that  $\alpha = \mathbf{e}_1 - \mathbf{e}_2$ . Now it is easy to compute  $\mathcal{E}$  (cf. [Dol]). For example, for  $d = 6$  we have  $\mathcal{E} = \{\mathbf{e}_3, \mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2\}$ , and for  $d = 5$  we have  $\mathcal{E} = \{\mathbf{e}_4, \mathbf{e}_4, \mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2, \mathbf{h} - \mathbf{e}_3 - \mathbf{e}_4\}$ . Other cases are similar. For  $d = 1$  we also can observe that  $\mathcal{E} = \Delta'' + K_S$  and apply Corollary 7.1.3.  $\square$

Below we describe del Pezzo threefolds  $X$  with  $r(X) + d(X) = 9$  explicitly and give examples. These threefolds were studied extensively in classical literature (see, e.g., [SR85, ch VIII, §2]). We assume that  $X$  is singular (otherwise  $X \simeq \mathbb{P}^3$ ,  $V_7$ , or  $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ ).

**7.3. Sextic del Pezzo threefold.** Let  $X \subset \mathbb{P}^2 \times \mathbb{P}^2$  be given by the equation  $x_1y_1 + x_2y_2 = 0$ . Then  $X$  is a del Pezzo threefold with  $d(X) = 6$  and  $r(X) = 3$ . The singular locus consists of one node.

**7.4. Quintic del Pezzo threefold (cf. [Tod30]).** Let  $X \subset \text{Gr}(2, 5)$  be an intersection of three general Schubert subvarieties of codimension one. Then  $X$  is a del Pezzo threefold with  $d(X) = 5$  and  $r(X) = 4$ . The singular locus consists of three nodes.

**7.5. Quartic del Pezzo threefold.** Let  $X \subset \mathbb{P}^5$  be an intersection of two quadrics having 6 isolated singular points. Then in some coordinate

system  $X$  can be given by the equations

$$(7.5.1) \quad x_1^2 - x_2^2 = x_3^2 - x_4^2 = x_5^2 - x_6^2.$$

In [SR85, ch VIII, 2.31] this variety is called the *tetrahedral quartic threefold*. By Corollary 10.6.2  $r(X) = 5$ . The variety  $X$  contains 8 planes

$$\Pi_{\epsilon_1, \epsilon_2, \epsilon_3} = \{x_1 + \epsilon_1 x_2 = x_3 + \epsilon_2 x_4 = x_5 + \epsilon_3 x_6 = 0\},$$

where  $\epsilon_i = \pm 1$ . Clearly,

$$\dim \Pi_{\epsilon_1, \epsilon_2, \epsilon_3} \cap \Pi_{\epsilon'_1, \epsilon'_2, \epsilon'_3} = -1 + \frac{1}{2} \sum |\epsilon_i + \epsilon'_i|.$$

Therefore, for each plane  $\Pi = \Pi_{\epsilon_1, \epsilon_2, \epsilon_3}$  there is exactly 3 planes  $\Pi'$  such that  $\Pi \cap \Pi'$  is a point and exactly 3 planes  $\Pi'$  such that  $\Pi \cap \Pi'$  is a line. Note that there are two 4-tuples of planes such that planes in each tuple meet each other only by subsets of dimension  $\leq 0$ :

$$\{\Pi_{+++}, \Pi_{+--}, \Pi_{-+-}, \Pi_{--+}\}, \quad \{\Pi_{---}, \Pi_{-++}, \Pi_{+-+}, \Pi_{+--}\}.$$

The involution

$$\tau : (x_1, x_2, x_3, x_4, x_5, x_6) \longmapsto (x_1, -x_2, x_3, -x_4, x_5, -x_6)$$

interchanges these 4-tuples. Hence  $\tau$  induces a birational (cubo-cubic) involution on  $\mathbb{P}^3$ . In [Hud27, Ch. XIV, §14, P. 301] it is denoted by  $T_{\text{tet}}$ . Note however that  $\text{Cl}(X)^\tau \not\cong \mathbb{Z}$ , i.e.  $X$  is not  $\tau$ -minimal.  $X$  is minimal with respect to the whole automorphism group.

**7.6. Segre cubic.** If  $d(X) = 3$ , then  $X$  can be given by

$$(7.6.1) \quad X = X_3^s = \left\{ \sum_{i=1}^6 x_i = \sum_{i=1}^6 x_i^3 = 0 \right\} \subset \mathbb{P}^4 \subset \mathbb{P}^5.$$

This cubic satisfies many remarkable properties (see [SR85, ch VIII, 2.32]) and is called the *Segre cubic*. For example, any cubic hypersurface in  $\mathbb{P}^4$  has at most ten isolated singular points, this bound is sharp and achieved exactly for the Segre cubic (up to projective isomorphism). The symmetric group  $\mathfrak{S}_6$  acts on  $X_3^s$  in the standard way. Moreover, by Corollaries 7.1.3 and 7.1.4 we see that  $\text{Aut}(X_3^s) = \mathfrak{S}_6$ , so the natural map  $\text{Aut}(X_3^s) \rightarrow W(\Delta'')$  is an isomorphism.

**7.7. Quartic double solid.** Let  $X$  be a del Pezzo threefold of degree 2. Let  $\phi : X \rightarrow \mathbb{P}^3$  be the half-anticanonical map. Then  $\phi$  is a double cover whose branch locus  $B \subset \mathbb{P}^3$  is a quartic having 16 singular points. It is well-known that such a quartic must be a Kummer surface, so the singularities of  $B$  and  $X$  are at worst nodes [Hud05], [Nik75] (see also [Jes16]). The singular points of  $X$  correspond to 15 lines  $L_{ij}$  passing through pairs of points  $P_i, P_j$  and one twisted cubic passing through all points  $P_1, \dots, P_6$ . The threefold  $X$  contains 32 planes [SR85, ch VIII,



2.33]. For each such a plane  $\Pi$  the image  $\pi(\Pi)$  is a plane touching  $B$  along a conic.

**7.7.1. Example.** Let  $B \subset \mathbb{P}^3$  be a surface given by the equation  $x_0^4 + x_1^4 + x_2^4 + x_3^4 - 4x_0x_1x_2x_3 = 0$ . Then the singular locus of  $S$  consists of 16 isolated points which are simple nodes. A double cover  $X \rightarrow \mathbb{P}^3$  branched along  $B$  is a del Pezzo threefold with  $d(X) = 2$  and  $r(X) = 7$ .

**7.8. Double Veronese cone.** Recall that  $X \simeq X_6 \subset \mathbb{P}(1^3, 2, 3)$ . The projection from  $(0, 0, 0, 0, 1)$  induces a double cover  $X \rightarrow \mathbb{P}(1^3, 2)$  with branch divisor  $B = B_6 \subset \mathbb{P}(1^3, 2)$ . Assume for simplicity that the singularities of  $X$  are at worst nodes. Then  $B$  is a surface having exactly 28 points of type  $A_1$ . Conversely if  $B \subset \mathbb{P}(1^3, 2)$  is a surface of degree 6 whose singularities are exactly 28 points of type  $A_1$ , then the double cover of  $\mathbb{P}(1^3, 2)$  branched at  $B$  is a del Pezzo threefold with  $d(X) = 1$  and  $r(X) = 8$ . We refer to [DO88] for more detailed treatment and more references.

**7.8.1. Example.** Let  $C \subset \mathbb{P}^2$  is given by the equation  $f = x_1^4 + x_2^4 + x_3^4$ . Then the dual curve  $C^*$  is given by  $f^* = (x_1^4 + x_2^4 + x_3^4)^3 - 27x_1^4x_2^4x_3^4$ . It is easy to check that the discriminant of the polynomial  $h(t) = t^3 - (x_1^4 + x_2^4 + x_3^4)t + 2x_1^2x_2^2x_3^2$  is equal to  $4f^*$ . The last polynomial defines a surface  $B \subset \mathbb{P}(1^3, 2)$  of degree 6 having 28 singular points.

**7.9. Corollary.** Let  $X$  be a del Pezzo threefold such that  $d(X) + r(X) = 9$  and  $d(X) \neq 5, 6, 7$ . Then  $X$  is a  $G$ -del Pezzo threefold with respect to some (geometric) group  $G$ .

## 8. DEL PEZZO THREEFOLDS WITH $r(X) = 8 - d(X)$

Let, as above,  $V_6 \subset \mathbb{P}^7$  be a smooth del Pezzo threefold of degree 6 and let  $f_i : V_6 \rightarrow \mathbb{P}^2$ ,  $i = 1, 2$  be  $\mathbb{P}^1$ -bundles. We say that points  $P_1, \dots, P_n \in V_6$  are in *general position* if so are the points  $f_i(P_1), \dots, f_i(P_n) \in \mathbb{P}^2$  for  $i = 1$  and  $2$ .

**8.1. Theorem.** Let  $X$  be a del Pezzo threefold with  $r(X) + d(X) = 8$ . Then

- (i)  $X$  can be obtained by applying construction (3.9.1) to  $V_6 \subset \mathbb{P}^7$  where  $\sigma$  is the blowup of from  $n := 6 - d(X)$  points  $P_1, \dots, P_n \in V_6$  in general position.
- (ii) Singular points of  $X$  are images of proper transforms of
  - (a) curves of bidegree  $(0, 1)$  and  $(1, 0)$  passing through one of the points  $P_i$ ;
  - (b) curves of bidegree  $(1, 1)$  passing through two of the points  $P_i$ ;
  - (c) curves of bidegree  $(2, 2)$  passing through four of the points  $P_i$ ;

- (d) (only for  $d(X) = 1$ ) curves of bidegree  $(2, 3)$  and  $(3, 2)$  passing through all the points  $P_i$ .
- (iii) If all the singularities of  $X$  are nodes, then  $s(X) = 27, 15, 9, 5, 2, 0$  in cases  $d(X) = 1, 2, 3, 4, 5, 6$ , respectively.
- (iv) If  $d(X) \geq 2$ , then all the singularities of  $X$  are nodes.

Conversely, assume that  $X$  is a del Pezzo threefold whose singularities are at worst nodes and assume that  $s(X) = 27, 15, 9, 5, 2, 0$  in cases  $d(X) = 1, 2, 3, 4, 5, 6$ , respectively. Then  $d(X) + r(X) = 8$ .

*Proof.* Run construction (3.9.1) so that  $n$  is maximal possible. On the last step we get a primitive weak del Pezzo threefold  $\bar{X}$  with  $\rho(\bar{X}) = 8 - d(\bar{X}) < 8$ . Moreover, if  $\rho(\bar{X}) = 3$ , then  $n = 0$ ,  $\rho(\hat{X}) = r(X) = 3$ , and  $d(X) = 5$ . This is impossible by Theorem 4.2. Therefore,  $\rho(\bar{X}) = 2$  and  $d(\bar{X}) = 6$ . By Theorem 5.2 we have only one possibility:  $\bar{X} \simeq V_6$ .  $\square$

**8.1.1. Corollary.** *Let  $X$  be a del Pezzo threefold with  $r(X) + d(X) = 8$ . If  $d(X) \geq 5$ , then  $X$  is unique up to isomorphism. There are exactly two isomorphism classes of del Pezzo threefolds with  $d(X) = r(X) = 4$ .*

*Proof.* Indeed, in the case  $d(X) = 4$  two non-isomorphic del Pezzo threefolds  $X$  are obtained by blowing up a couple of points corresponding to flags  $(L_1, P_1), (L_2, P_2) \in F(\mathbb{P}^2) = V_6$  such that either  $L_1 \cap L_2 \neq P_i$  or  $L_1 \cap L_2 = P_i$ .  $\square$

Similar to Theorem 7.2 one can prove the following.

**8.2. Theorem.** *Let  $X$  be a del Pezzo threefold with  $r(X) + d(X) = 8$ . Let  $\Pi \subset X$  is a plane, let  $\hat{\Pi} \subset \hat{X}$  be its proper transform, and let  $\bar{\Pi} = \sigma(\hat{\Pi}) \subset \bar{X} = V_6$ . Then  $\bar{\Pi}$  is of one of the following types:*

- (i)  $\bar{\Pi}$  is one of the points  $P_i$ ,  $\hat{\Pi}$  is  $\sigma$ -exceptional;
- (ii)  $f_j(\bar{\Pi})$  is a line for  $j = 1$  or  $2$ , and  $\bar{\Pi}$  contains two of the points  $P_i$ ;
- (iii)  $\bar{\Pi}$  is an element of  $|- \frac{1}{2}K_{V_6}|$  passing through four of the points  $P_i$  so that one of them is a double point;
- (iv) (only for  $d(X) = 1$ )  $f_j(\bar{\Pi})$  is a conic for  $j = 1$  or  $2$ , and  $\bar{\Pi}$  contains all the points  $P_i$ ;
- (v) (only for  $d(X) = 1$ )  $\bar{\Pi}$  is an element of  $|-K_{V_6}|$  passing through all of the points  $P_i$  so that all of them are double points and one of them is triple;
- (vi) (only for  $d(X) = 1$ )  $\bar{\Pi}$  is an element of  $|-K_{V_6} - f_j^* \mathcal{O}_{\mathbb{P}^2}(1)|$ , where  $j = 1$  or  $2$ , passing through all of the points  $P_i$  so that three of them are double points.

The number of planes on  $X$  is given by the following table:

|        |   |   |   |   |    |    |
|--------|---|---|---|---|----|----|
| $d(X)$ | 6 | 5 | 4 | 3 | 2  | 1  |
| $p(X)$ | 0 | 1 | 4 | 9 | 20 | 72 |

**8.2.1. Corollary.** *Let  $X$  be a del Pezzo threefold with  $r(X) = 8 - d(X)$  and  $d(X) \leq 5$ . Then in some standard basis of  $\text{Pic}(S)$  the image of  $\iota^* : \text{Cl}(X) \rightarrow \text{Pic}(S)$  is a sublattice orthogonal to roots  $\mathbf{e}_1 - \mathbf{e}_2$ ,  $\mathbf{e}_2 - \mathbf{e}_3 \in \Delta$ , i.e.  $\Delta' = \{\pm \mathbf{e}_1 \mp \mathbf{e}_2, \pm \mathbf{e}_2 \mp \mathbf{e}_3, \pm(\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3)\}$ . Moreover,  $\Delta''$  is of type  $E_6$ ,  $D_5$ ,  $2A_2$ ,  $2A_1$  in cases  $d(X) = 1, 2, 3, 4$ , respectively.*

Similar to Corollary 7.1.4 (ii) we have.

**8.2.2. Corollary.** *Let  $X$  be a del Pezzo threefold with  $r(X) = 8 - d(X)$  and  $d(X) \leq 2$ . If  $\mathbb{k}$  is algebraically closed (i.e. we are in the geometric case), then the map  $G \rightarrow \text{Aut}(\Delta'')$  is an embedding.*

Now we give some examples.

**8.3. Quintic del Pezzo threefold (cf. [Tod30]).** Let  $X \subset \text{Gr}(2, 5)$  be an intersection of two general Schubert subvarieties of codimension one and one general hyperplane section. Then  $X$  is a del Pezzo threefold with  $d(X) = 5$  and  $r(X) = 3$ . The singular locus consists of two nodes.

**8.3.1. Corollary (cf. [Tod30], [Fuj86]).** *Let  $X$  be a del Pezzo threefold of degree 5. Then the singularities of  $X$  are at worst nodes and one of the following holds:*

- (i)  $X \simeq V_5$ , a smooth del Pezzo quintic threefold;
- (ii)  $s(X) = 1$ ,  $r(X) = 2$ ,  $p(X) = 0$ , and  $X$  is of type 5.4.2;
- (iii)  $s(X) = 2$ ,  $r(X) = 3$ ,  $p(X) = 1$ , and  $X$  is of type 7.4;
- (iv)  $s(X) = 3$ ,  $r(X) = 4$ ,  $p(X) = 4$ , and  $X$  is of type 8.3.

*Proof.* Assertions (iii) or (iv) follows by the results of this and previous sections. If  $r(X) = 2$ , then we have case (ii) by Theorem 5.2. Finally, if  $X$  is factorial, then it is smooth by Corollary 5.3.  $\square$

**8.4. Quartic del Pezzo threefold.** Let  $X \subset \mathbb{P}^5$  be given by the equations

$$x_1^2 + x_1x_3 + x_2x_5 = x_1x_3 + x_3^2 + x_4x_6 = 0.$$

Then  $X$  is a del Pezzo threefold of degree 4 containing exactly 5 nodes. By Corollary 10.6.2  $r(X) \geq 4$ . On the other hand,  $X$  is not of type 7.5 because  $s(X) < 6$ . Hence  $r(X) = 4$ .

**8.5. Cubic hypersurface.** Let  $X \subset \mathbb{P}^5$  be given by the equation

$$x_1x_2\ell(x_1, \dots, x_5) + (x_3x_4 + x_1x_2)x_5 = 0,$$

where  $\ell$  is a general linear form. Then  $X$  is a cubic del Pezzo threefold with  $s(X) = 9$ ,  $r(X) = 5$ , and  $p(X) = 9$  (cf. [FW, J14]).

**8.6. Quartic double solid.** Let  $Y$  be a hypersurface in  $\mathbb{P}^4$  given by  $\{s_1 = 4s_4 - s_2^2 = 0\} \subset \mathbb{P}^5$ , where  $s_k = \sum x_i^k$ . This famous hypersurface is called *Igusa quartic*. The singular locus of  $Y$  consists of 15 lines. Consider a general hyperplane section  $B := Y \cap \mathbb{P}^3$ . Then  $B$  is a quartic having 15 nodes (cf. [Jes16]). Let  $X \rightarrow \mathbb{P}^3$  be a double cover with branch divisor  $B$ . Then  $X$  is a del Pezzo threefold of degree 2 with  $s(X) = 15$  and  $r(X) = 6$ .

## 9. $G$ -DEL PEZZO THREEFOLDS

**9.1.** In this section we prove Theorem 1.7. We use notation of 6.1. Furthermore we assume that  $X$  is a  $G$ -del Pezzo threefold. Thus  $\mathrm{Cl}(X)^G \simeq \mathbb{Z}$ . By Theorem 5.2 we may assume that  $r(X) \geq 3$ .

**9.2. Lemma.** *In the above notation, if  $d(X) \geq 5$ , then  $X \simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ .*

*Proof.* Assume that  $X \not\simeq \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ . Then  $X$  is singular and  $d(X) = 6$  or 5 by Theorems 3.2 and 3.5.

Consider the case  $d(X) = 6$ . Since  $r(X) \geq 3$ , our  $X$  is described in 7.3. Then  $X$  contains exactly two planes  $\Pi_1, \Pi_2$  and the divisor  $\Pi_1 + \Pi_2$  is  $G$ -invariant. Hence  $\Pi_1 + \Pi_2 \sim aS$  for some positive integer  $a$ . Comparing degrees we get  $2 = 6a$ , a contradiction.

Now let  $d(X) = 5$ . By Lemma 3.3 we may assume that  $X$  is not factorial. In this situation,  $X$  is imprimitive. The same arguments as above show that the number of planes on  $X$  in any  $G$ -orbit must be divisible by 5. This contradicts Corollary 8.3.1.  $\square$

**9.3.** From now on we assume that  $d(X) \leq 4$ . By Theorem 4.2 we may assume that  $X$  is imprimitive. Let  $S \in |-\frac{1}{2}K_X|$  be a general member. Let  $n := \mathrm{rk} \Delta = 9 - d(X)$ .

**9.3.1. Lemma.** *If in the above notation  $d(X) \leq 4$ , then  $X$  contains at least two planes  $\Pi_1, \Pi_2$  such that  $\dim \Pi_1 \cap \Pi_2 \leq 0$ .*

*Proof.* Since  $X$  is imprimitive, it contains at least one plane  $\Pi_1$ . Let  $\Pi_1, \dots, \Pi_l$  be its orbit. Since  $\mathrm{Cl}(X)^G = \mathbb{Z} \cdot S$ ,  $k \geq 4$ . If  $\dim \Pi_i \cap \Pi_j \geq 1$  for all  $i, j$ , the linear span of  $\Pi_1, \dots, \Pi_k$  is three-dimensional and so  $X$  cannot be an intersection of quadrics.  $\square$

First we consider the case  $\Delta'' = \emptyset$ .

**9.4. Proposition.** *If in the above notation  $\Delta'' = \emptyset$ , then  $d(X) = 3$ ,  $r(X) = 3$ ,  $p(X) = 3$ , and  $X$  is a projection of a del Pezzo threefold  $Y = Y_4 \subset \mathbb{P}^5$  of type (5.2.8) from a point. If moreover the singularities of  $X$  are at worst nodes, then by [FW],  $X$  is of type J11 or J12, and  $6 \leq s(X) \leq 7$ .*

*Proof.* Let  $\mathbf{e}_{n-m+1}, \dots, \mathbf{e}_n$  correspond to blowups  $\sigma$ . If  $m > 1$ , then  $\mathbf{e}_{n-1} - \mathbf{e}_n \in \Delta''$ . Thus,  $m = 1$ ,  $d(\bar{X}) = d(X) + 1$ , and we may assume that every two planes on  $X$  meet each other by a subset of dimension 1. Therefore,  $r(\bar{X}) = 2$  and  $r(X) = 3$ . By Lemma 9.3.1  $d(X) \leq 3$ . Therefore,  $d(\bar{X}) \leq 4$ , and  $8 \geq n \geq 6$ .

Consider the case where  $f : \bar{X} \rightarrow Z = \mathbb{P}^2$  is a  $\mathbb{P}^1$ -bundle. Then by Theorem 6.2 we may assume that the vectors  $\mathbf{e}_1 - \mathbf{e}_2, \dots, \mathbf{e}_{n-2} - \mathbf{e}_{n-1}$  form a basis of  $\Delta'$ . We have:

$$\begin{aligned} n = 6, d(X) = 3 &\implies 2\mathbf{h} - \sum \mathbf{e}_i \in \Delta'', \\ n = 7, d(X) = 2 &\implies 2\mathbf{h} + \mathbf{e}_7 - \sum \mathbf{e}_i \in \Delta'', \\ n = 8, d(X) = 1 &\implies 3\mathbf{h} - \mathbf{e}_8 - \sum \mathbf{e}_i \in \Delta''. \end{aligned}$$

Thus, in all cases we have  $\Delta'' \neq \emptyset$ , a contradiction.

Consider the case where  $f : \bar{X} \rightarrow Z = \mathbb{P}^1$  is a quadric bundle. Then  $d(X) = d(\bar{X}) - 1 = 1$  or  $3$  by Theorem 5.2. Again by Theorem 6.2 we may assume that vectors

$$\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3, \quad \mathbf{e}_2 - \mathbf{e}_3, \dots, \mathbf{e}_{n-2} - \mathbf{e}_{n-1}$$

form a basis of  $\Delta'$ . If  $n = 8$ , then  $3\mathbf{h} - \mathbf{e}_8 - \sum \mathbf{e}_i \in \Delta''$ , a contradiction. Therefore,  $d(X) = 3$  and  $\bar{X}$  is of type (5.2.8). Thus  $X$  is a cubic in  $\mathbb{P}^4$ . Since for any two planes  $\Pi_i, \Pi_j \subset X$  we have  $\dim \Pi_i \cap \Pi_j \geq 1$ , all the planes on  $X$  are contained in one hyperplane. Hence  $p(X) = 3$ . By Proposition 10.6  $s(X) \leq 7 - h^{1,2}(\hat{X})$ . If the singularities of  $X$  are at worst nodes, then  $X$  is of type J11 or J12 by [FW].  $\square$

**9.4.1. Example.** Consider the cubic  $X \subset \mathbb{P}^4$  given by the equation

$$x_1 x_2 x_3 + x_0 (\lambda x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4^2) = 0.$$

Then  $X$  has 6 (resp. 7) nodes if  $\lambda \neq 0$  (resp.  $\lambda = 0$ ). It is easy to see that  $X$  contains at least 3 planes, so  $r(X) \geq 3$ . By [FW] we have  $r(X) = 3$  and  $p(X) = 3$ . The symmetric group  $\mathfrak{S}_3$  acts on  $X$  by permutations of  $x_1, x_2, x_3$  so that  $X$  is a  $G$ -Fano threefold.

**9.5.** Now we assume that  $\Delta'' \neq \emptyset$ . Then  $\Delta''$  is a  $G$ -invariant root subsystem in  $\Delta$ . By the results of §7 and §8 we may assume that  $r(X) \leq 7 - d(X)$ . Further, by Lemma 9.3.1  $d(X) \leq 3$ .

**9.6.** Consider the case  $d(X) = 3$ . There are only the following possibilities:

**9.6.1.**  $d(\bar{X}) = 4$ ,  $r(\bar{X}) = 2$ ,  $\bar{X}$  is of type (5.2.8),  $r(X) = 3$ . Then  $\Delta'$  is described in 6.3.2: it is of type  $D_4$  and generated by  $\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$ ,  $\mathbf{e}_2 - \mathbf{e}_3$ ,  $\mathbf{e}_3 - \mathbf{e}_4$ ,  $\mathbf{e}_4 - \mathbf{e}_5$ . Any root  $\alpha \in \Delta$  has the form  $\alpha = \pm(\mathbf{e}_i - \mathbf{e}_j)$ ,  $\pm(\mathbf{h} - \mathbf{e}_i - \mathbf{e}_j - \mathbf{e}_k)$  or  $\pm(2\mathbf{h} - \sum \mathbf{e}_i)$  (see e.g., [Man86, ch. 4, 3.7]). Since  $\iota^* \text{Cl}(X) = \Delta'^{\perp}$ , we get  $\Delta'' = \emptyset$ , a contradiction.

**9.6.2.**  $d(\bar{X}) = 5$ ,  $r(\bar{X}) = 1$ ,  $\bar{X} = V_5$ ,  $r(X) = 3$ . Similarly,  $\Delta'$  is of type  $A_4$  and generated by  $\mathbf{h} - \mathbf{e}_1 - \mathbf{e}_2 - \mathbf{e}_3$ ,  $\mathbf{e}_1 - \mathbf{e}_2$ ,  $\mathbf{e}_2 - \mathbf{e}_3$ ,  $\mathbf{e}_3 - \mathbf{e}_4$ . In this case,  $\Delta'' = \{\pm(\mathbf{e}_5 - \mathbf{e}_6)\}$ . It is easy to see that the group  $G$  permutes elements  $\mathbf{e}_5$ ,  $\mathbf{e}_6 \in \iota^* \text{Cl}(X)$ . But then the class of  $\mathbf{e}_5 + \mathbf{e}_6$  must be  $G$ -invariant, so it is proportional to  $-K_S$ , a contradiction.

**9.6.3.**  $d(\bar{X}) = 5$ ,  $r(\bar{X}) = 2$ ,  $\bar{X}$  is of type (5.2.5),  $r(X) = 4$ . Similarly,  $\Delta'$  is of type  $A_3$  and generated by  $\mathbf{e}_1 - \mathbf{e}_2$ ,  $\mathbf{e}_2 - \mathbf{e}_3$ ,  $\mathbf{e}_3 - \mathbf{e}_4$ . Then  $\Delta'' = \{\pm(2\mathbf{h} - \sum \mathbf{e}_i), \pm(\mathbf{e}_5 - \mathbf{e}_6)\}$ . There is a unique element (class of a line on  $S$ )  $\mathbf{x} \in (\Delta' + \Delta'')^\perp$  such that  $\mathbf{x}^2 = K_X \cdot \mathbf{x} = -1$ :

$$\mathbf{x} = \mathbf{h} - \mathbf{e}_5 - \mathbf{e}_6.$$

But then  $x \in \iota^* \text{Cl}(X)$  and  $x$  must be  $G$ -invariant, a contradiction.

**9.7.** Finally we consider cases  $d(X) \leq 2$ . According to Remark 3.4.1 any del Pezzo threefold with  $d(X) \leq 2$  is automatically  $G$ -del Pezzo. Thus all the possibilities for  $\bar{X}$  with  $2 \leq d(\bar{X}) \leq 5$  and  $r(\bar{X}) \leq 2$  do occur (recall that  $3 \leq r(X) \leq 7 - d(X)$ ):

- $\bar{X} = V_5 \implies d(X) \leq 2$ ,  $\Delta' \simeq A_4$ ;
- $\bar{X} = V_4 \implies d(X) \leq 2$ ,  $\Delta' \simeq D_5$ ;
- $\bar{X} = V_3 \implies d(X) \leq 2$ ,  $\Delta' \simeq E_6$ ;
- $\bar{X}$  is of type (5.2.2)  $\implies d(X) = 1$ ,  $\Delta' \simeq A_6$ ;
- $\bar{X}$  is of type (5.2.3)  $\implies d(X) \leq 2$ ,  $\Delta' \simeq A_5$ ;
- $\bar{X}$  is of type (5.2.5)  $\implies d(X) \leq 2$ ,  $\Delta' \simeq A_3$ ;
- $\bar{X}$  is of type (5.2.7)  $\implies d(X) = 1$ ,  $\Delta' \simeq D_6$ ;
- $\bar{X}$  is of type (5.2.8)  $\implies d(X) \leq 2$ ,  $\Delta' \simeq D_4$ .

The number of planes can be found by using Lemma 3.7.3 and direct computations.

**9.7.1. Example.** Let  $X \subset \mathbb{P}(1^4, 2)$  is given by the equation

$$y^2 = x_1 x_2 x_3 x_4 + \lambda(x_1^2 + x_2^2 + x_3^2 + x_4^2)^2$$

where  $\lambda$  is a constant. Then  $X$  has exactly 12 nodes and contains 8 planes. By Corollary 10.6.2  $r(X) \geq 3$ . Further, by our classification  $X$  is of type  $22^\circ$ .

More examples of del Pezzo threefolds with  $d(X) = 2$  can be constructed similarly by writing down explicit equations (cf. [Jes16]).

## 10. APPENDIX: NUMBER OF SINGULAR POINTS OF FANO THREEFOLDS

**10.1. Definition.** Let  $V \ni P$  be a threefold terminal Gorenstein (=isolated cDV) singularity. We say that  $V \ni P$  is  $r$ -nondegenerate (resolution nondegenerate) if there is a resolution

$$\sigma : V_m \xrightarrow{\sigma_m} \cdots \xrightarrow{\sigma_2} V_1 \xrightarrow{\sigma_1} V = V_0,$$

where each  $\sigma_i$  is a blowup of a singular point  $P_{i-1} \in V_{i-1}$ . Such a resolution  $\sigma$  is called *standard*. In this situation, all varieties  $V_i$  also have only isolated cDV singularities. If furthermore each  $\sigma_i$ -exceptional divisor  $E_i \subset V_i$  is irreducible, then we say that  $V \ni P$  is *rs-nondegenerate* (strongly resolution nondegenerate).

Denote  $\lambda(V, P) := m$  and let  $\nu(V, P)$  be the number of  $\sigma$ -exceptional divisors. Thus  $\lambda(V, P) \leq \nu(V, P)$  and the equality holds if and only if  $V \ni P$  is rs-nondegenerate.

**10.2. Remark.** Let  $V \ni P$  be a threefold terminal Gorenstein point and let  $\sigma_1 : V_1 \rightarrow V$  be the blowup of  $P$ . Since  $V \ni P$  is a hypersurface singularity, we have an (analytic) embedding  $V_1 \subset \tilde{\mathbb{C}}^4$ , where  $\tilde{\sigma}_1 : \tilde{\mathbb{C}}^4 \rightarrow \mathbb{C}^4$  is the blowup of the origin. Let  $D := \tilde{\sigma}_1^{-1}(P)$  be the exceptional divisor. Then  $D \simeq \mathbb{P}^3$ . Since  $V \ni P$  is a singularity of multiplicity 2, the intersection  $D \cap V_1$  is a quadric in  $\mathbb{P}^3$  (possibly reducible or non-reduced). If  $D \cap V_1$  irreducible, then  $V_1$  is either smooth or has (a unique) terminal singularity. Moreover, the above arguments show that  $2\lambda(V, P) \geq \nu(V, P)$ .

**10.3. Proposition.** Let  $(V \ni 0) \subset \mathbb{C}^4$  be a singularity given by  $t^2 = \phi(x, y, z)$ , where  $\phi = 0$  is an equation of a Du Val singularity. Then  $V \ni 0$  is *r-nondegenerate*. Moreover, if  $\phi = 0$  defines a singularity of type  $A_n$ , then  $V \ni 0$  is *rs-nondegenerate*.

*Proof.* Direct computation. □

**10.3.1. Corollary.** Let  $X$  be a del Pezzo threefold with  $d(X) \leq 2$ . Assume that the branch divisor  $B$  of the double cover  $\varphi : X \rightarrow \mathbb{P}(1^3, 2)$  (resp.  $\varphi : X \rightarrow \mathbb{P}^3$ ) has only Du Val singularities (see 3.4.1). Then the singularities of  $X$  are *r-nondegenerate*. If moreover  $B$  has only singularities of type  $A$ , then the singularities of  $X$  are *rs-nondegenerate*.

**10.4.** Let  $W$  be a smooth projective fourfold and let  $V \subset W$  be an effective divisor. Define

$$\beta(W, V) := c_3(W) \cdot V - c_2(W) \cdot V^2 + c_1(W) \cdot V^3 - V^4.$$

If  $V$  is smooth then  $\beta(W, V)$  coincides with  $\deg c_3(V) = \text{Eu}(V)$ , the topological Euler number.

**10.5. Lemma.** In the above notation let  $P \in V$  be a singular point, let  $\sigma : \tilde{W} \rightarrow W$  be the blowup of  $P$ , and let  $\tilde{V} \subset \tilde{W}$  be the proper transform of  $V$ . Then  $\beta(\tilde{W}, \tilde{V}) = \beta(W, V) + 4$ .

*Proof.* Let  $R = \sigma^{-1}(P)$  be the exceptional divisor in  $\tilde{W}$ , and let  $E = R \cap \tilde{V}$  be the exceptional divisor in  $\tilde{V}$ . We have

$$\begin{aligned}\tilde{V} &\sim \sigma^*V - 2E, & c_3(\tilde{W}) &= \sigma^*c_3(W) + 2E^3, \\ c_2(\tilde{W}) &= \sigma^*c_2(W) + 2E^2, & c_1(\tilde{W}) &= \sigma^*c_1(W) - 3E.\end{aligned}$$

Using the equality  $c(\tilde{V}) = c(\tilde{W}) \cdot c(N_{\tilde{V}/\tilde{W}})^{-1}$ , we get

$$\begin{aligned}\beta(\tilde{W}, \tilde{V}) &= c_3(\tilde{W}) \cdot \tilde{V} - c_2(\tilde{W}) \cdot \tilde{V}^2 + c_1(\tilde{W}) \cdot \tilde{V}^3 - \tilde{V}^4 = \\ &= (\sigma^*c_3(W) + 2E^3) \cdot (\sigma^*V - 2E) - (\sigma^*c_2(W) + 2E^2) \cdot (\sigma^*V - 2E)^2 + \\ &\quad + (\sigma^*c_1(W) - 3E) \cdot (\sigma^*V - 2E)^3 - (\sigma^*V - 2E)^4 = \beta(W, V) - 4E^4.\end{aligned}$$

□

**10.6. Proposition.** *Let  $X$  be a Gorenstein Fano threefold whose singularities are  $r$ -nondegenerate terminal points. Assume that*

(\*)  *$X$  can be embedded to a smooth fourfold so that a general member  $X' \in |X|$  is smooth.*

*Then*

$$\begin{aligned}\sum'_{P \in X} \lambda(X, P) &\leq \sum_{P \in X} (2\lambda(X, P) - \nu(X, P)) = \\ &= r(X) - \rho(X) + h^{1,2}(X') - h^{1,2}(\hat{X}),\end{aligned}$$

where  $\hat{X} \rightarrow X$  is the standard resolution and the first sum runs through all  $rs$ -nondegenerate points  $P \in X$ .

*Proof.* Put  $\lambda := \sum_{P \in X} \lambda(X, P)$ . Thus

$$\begin{aligned}2 + 2\rho(\hat{X}) - 2h^{1,2}(\hat{X}) &= \text{Eu}(\hat{X}) = \beta(\hat{Y}, \hat{X}) = \beta(\hat{Y}, \hat{X}) + 4\lambda = \\ &= \text{Eu}(X') + 4\lambda = 2 + 2\rho(X') - 2h^{1,2}(X') + 4\lambda.\end{aligned}$$

Since  $\rho(\hat{X}) = r(X) + \sum \nu(X, P)$ , this gives the desired inequality. □

**10.6.1. Remark.** The condition (\*) is automatically satisfied if  $X$  is a del Pezzo threefold (see Theorem 3.4).

**10.6.2. Corollary.** *In the notation of 10.6 assume additionally that the singularities are  $rs$ -nondegenerate. Then*

$$|\text{Sing}(X)| \leq r(X) - \rho(X) + h^{1,2}(X') - h^{1,2}(\hat{X}).$$

*The equality holds, if all the singularities are nodes.*



## 11. CONCLUDING REMARKS AND OPEN QUESTIONS

We would like to propose the following open questions.

**11.1.** Give a complete birational classification of del Pezzo threefolds over  $\mathbb{C}$ . Non-trivial cases only are factorial del Pezzo threefolds of degree  $\leq 3$ . All other cases can be reduced to the above ones by using construction 3.9.1 (or birationally trivial). It is well-known that a three-dimensional cubic hypersurface with at worst cDV singularities is rational if and only if it is singular [CG72]. A general smooth (and, in some cases, factorial) del Pezzo threefold of degree  $\leq 2$  is not rational [AM72], [Bea77], [Tyu79].

**11.2.** Give a complete birational classification of del Pezzo threefolds over algebraically non-closed fields. Here is one example.

**11.2.1. Theorem.** *Let  $X$  be a smooth del Pezzo threefold of degree 5 over a field  $\mathbb{k}$ . Then  $X$  is  $\mathbb{k}$ -rational.*

*Proof.* Denote  $\bar{X} := X \otimes \bar{\mathbb{k}}$ . Let  $\Gamma := \Gamma(X)$  be the Hilbert scheme parameterizing the family of lines on  $X$ . It is known that  $\bar{\Gamma} := \Gamma \otimes \bar{\mathbb{k}} \simeq \mathbb{P}_{\bar{\mathbb{k}}}^2$  (see [Isk80a, Prop. 1.6, ch. 3], [FN89]). Moreover, lines with normal bundle  $N_{l/X} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(1)$  are parametrized by some conic  $C \subset \Gamma$  [FN89]. The conic  $C$  contains a point of degree  $\leq 2$ . Therefore, there is a line  $\ell \subset \Gamma$  defined over  $\mathbb{k}$ . Let  $H_\ell$  be the union of all lines  $L \subset X$  whose class is contained in  $\ell \subset \Gamma$ . Then  $H_\ell$  is an element of  $|- \frac{1}{2}K_X|$  defined over  $\mathbb{k}$  [Isk80a, Proof of Prop. 1.6, ch. 3]. In particular,  $\text{Pic}(X) = \mathbb{Z} \cdot \frac{1}{2}K_X$  and the linear system  $|- \frac{1}{2}K_X|$  defines an embedding  $X \subset \mathbb{P}_{\mathbb{k}}^6$ . A general pencil of hyperplane sections defines a structure of del Pezzo fibration of degree 5 on  $X$ . By [Man86, Ch. 4] the variety  $X$  is  $\mathbb{k}$ -rational.  $\square$

**11.3.** Describe automorphism groups of del Pezzo threefolds over an algebraically closed fields. Which of them are birationally rigid (cf. [CS09], [CS10])? These questions are very useful for applications to the classification of finite subgroups of Cremona group  $\text{Cr}_3(\mathbb{k})$  [Pro09], [Pro10].

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